Tutorial I
PDE-Constrained Optimization Under Uncertainty

Matthias Heinkenschloss

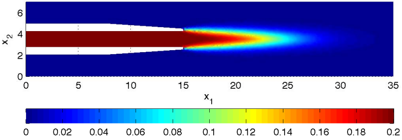
Department of Computational Applied Mathematics and Operations Research
Rice University, Houston, Texas
heinken@rice.edu

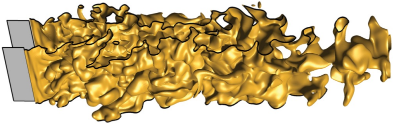
April 17, 2025

East Coast Optimization Meeting 2025
'Optimization and Digital Twins'
April 17 - 18, 2025
George Mason University, Arlington, Virginia (USA)

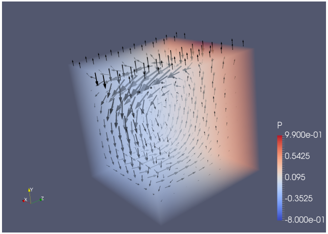
- PDE-constrained optimization arises in many engineering and science applications.
- Important task in digital-twins.

Nonlinear reaction advection diffusion



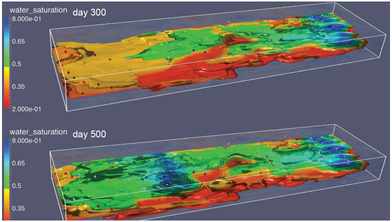


Flow control



(Thanks to Drew Kouri, Sandia National Labs)

Reservoir management



Direct field acoustic testing

PDE-Constrained Optimization

- Physics, ... described by systems of partial differential equations.
- Uncertainties in system.
- Optimization problems difficult to formulate and expensive to solve
  - Modeling of optimization problem (what is the objective, what are the controls, what physics need to be included, what is the decision time horizon,...)
  - Infinite dimensional problem structure impacts solution f discretized problem.
  - Expensive simulation.
  - Many control/design variables $\rightarrow$ derivative based methods
  - Optimize across all realizations of uncertainties in coefficients.
- Applications to design, control, parameter identification.
- Important task in digital twins.
- (First part of) Tutorial based on upcoming paper [Heinkenschloss and Kouri, 2025].

Outline

Elliptic Optimal Control Problem

- $D = (0,1) \times (0,1)$, control boundary $\partial D_c = \{0\} \times [0,1]$ and Neumann boundary $\partial D_n = \partial D \setminus \partial D_c$.
- Given parameterized coefficient functions

$$\kappa(x,\xi) = \begin{cases} \xi^{(1)}, & x \in [0,1] \times [0,0.6), \\ \xi^{(2)}, & x \in [0,1] \times [0.6,1], \end{cases} \qquad c(x) = \begin{pmatrix} 1 \\ \xi^{(3)} \end{pmatrix},$$

$$f(x,\xi) = 20 \exp\left(-\frac{(x_1 - \xi^{(4)})^2}{0.1} \right) \exp\left(-\frac{(x_2 - \xi^{(5)})^2}{0.1} \right)$$

with $\xi = \left(\xi^{(1)}, \xi^{(2)}, \xi^{(3)}, \xi^{(4)}, \xi^{(5)}\right)^\top$.

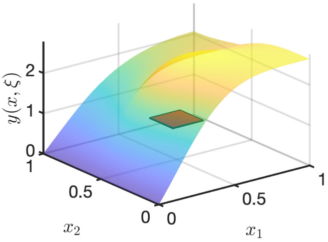
- Consider the linear elliptic PDE

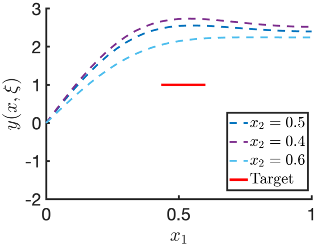
$$\begin{aligned} -\nabla \cdot (\kappa(x,\xi) \nabla y(x,\xi)) + c(x,\xi) \cdot \nabla y(x,\xi) &= f(x,\xi), && x \in D, \\ y(x,\xi) &= u(x), && x \in \partial D_c, \\ \big(\kappa(x,\xi) \nabla y(x,\xi)\big) \cdot n(x) &= 0, && x \in \partial D_n. \end{aligned}$$

- Initially set

$$\xi^{(1)} = 0.3, \quad \xi^{(2)} = 0.8, \quad \xi^{(3)} = 0.1, \quad \xi^{(4)} = 0.25, \quad \xi^{(5)} = 0.45.$$

- No-control ($u = 0$) PDE solution y and slices $y(x_1, x_2)$ of solution at $x_2 = 0.4, 0.5, 0.6$.



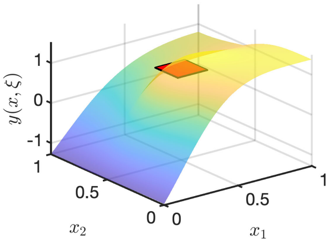


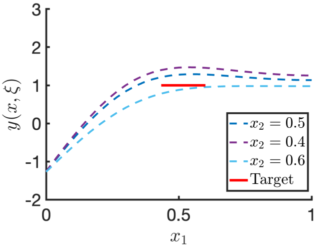
- Want control $u \in H^1(\partial D_c)$ such that PDE solution $y(u; \cdot, \xi)$ ideally is below a target, here 1, in obs. region $D_o = [0.4, 0.6] \times [0.4, 0.6]$.
- Optimal control problem

$$\min_{u \in H^1(\partial D_c)} \frac{1}{2} \int_{D_o} \big(y(u; x, \xi) - 1\big)_+^2 \, \mathrm{d}x + \frac{10^{-2}}{2} \|u\|^2_{H^1(\partial D_c)}, \qquad (1)$$

where $y(u; \cdot, \xi)$ is the PDE solution given u and ξ. $z_+ = \max\{z, 0\}$.

- Optimal state $y(u_*; \cdot, \xi)$

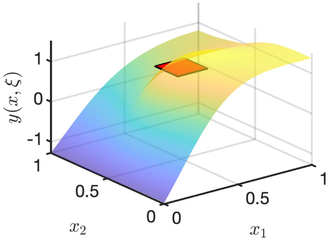


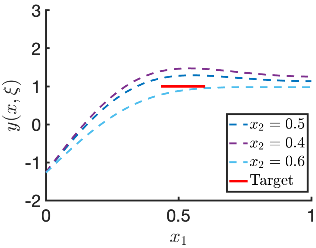


- Improves performance a lot at given parameter, but
- What if optimal control is applied with different parameters ξ?
- Interested in case where params. are not deterministic but are a realization of random variable (r.v.) $\boldsymbol{\xi} = \left(\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)}, \boldsymbol{\xi}^{(3)}, \boldsymbol{\xi}^{(4)}, \boldsymbol{\xi}^{(5)}\right)^\top$,

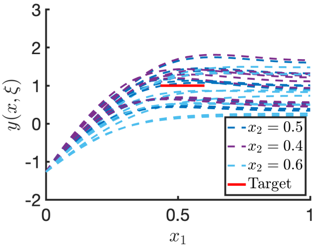
$$\boldsymbol{\xi}^{(1)} \sim U(0.2, 0.4), \quad \boldsymbol{\xi}^{(2)} \sim U(0.7, 0.9), \quad \boldsymbol{\xi}^{(3)} \sim U(0.0, 0.2),$$
$$\boldsymbol{\xi}^{(4)} \sim U(0.1, 0.4), \quad \boldsymbol{\xi}^{(5)} \sim U(0.3, 0.6).$$

- Optimal state $y(u_*; \cdot, \xi)$ with given parameter ξ





- What if optimal control is applied with different parameters ξ?
- Select 10 samples ξ_i of $\boldsymbol{\xi}$, compute $y(u_*; \cdot, \xi_i)$.
- Optimal control computed with fixed ξ can perform poorly if applied with different parameter values.
- Incorporate randomness in params. into computation of control.
- Interested in one control that is applied to all possible outcomes of $\boldsymbol{\xi}$. Must decide on control before uncertainty is revealed.



- Given a control $u \in H^1(\partial D_c)$, consider objective function

$$\frac{1}{2}\int_{D_o} \big(y(u;x,\xi) - 1\big)_+^2 \, \mathsf{d}x + \frac{10^{-2}}{2}\|u\|^2_{H^1(\partial D_c)}, \qquad (2)$$

  where $y(u;\cdot,\xi)$ solves PDE for $\xi \in \Xi$.

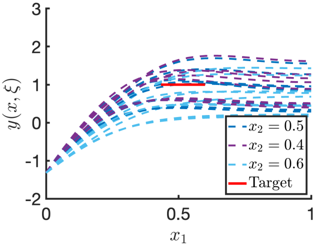
- Want control u that makes (2) small in some sense for all $\xi \in \Xi$.

- (2) is a function in $\xi \in \Xi$, need to quantify its size.

- One possibility is to take its expected value.:

$$\min_{u \in H^1(\partial D_c)} \int_\Xi \rho(\xi) \left[\frac{1}{2}\int_{D_o} \big(y(u;x,\xi) - 1\big)_+^2 \, \mathsf{d}x\right] \mathsf{d}\xi + \frac{10^{-2}}{2}\|u\|^2_{H^1(\partial D_c)}.$$

- Compute control u_* as solution of expected value min. problem.
- Compute corresponding state $y(u_*; x_1, x_2, \xi)$ at same 10 samples of $\boldsymbol{\xi}$ used before.
- Results in

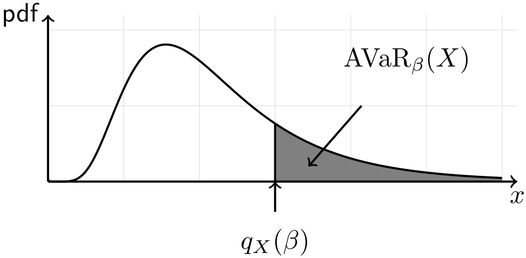


- Samples $y(u_*; \cdot, \xi)$ of the corresponding state can be significantly larger than target.
- Expected value minimization problem is called risk-neutral problem.

- It may be more harmful if $y(u;\cdot,\xi)$ exceeds the target by a lot, than if $y(u;\cdot,\xi)$ barely exceeds the target.
- Minimize (in some sense) $(1-\beta)\times 100\%$ of worst cases.
- Quantified using the *average value-at-risk* (AVaR).
- For a r.v. $X \in L^1$ and confidence level $\beta \in (0,1)$,

$$\text{AVaR}_\beta(X) := \min_{t\in\mathbb{R}} \left\{ t + \frac{1}{1-\beta}\mathbb{E}[\max\{0, X-t\}] \right\}.$$

- For continuous r.vs. X, $\text{AVaR}_\beta(X)$ is average of $(1-\beta)\times 100\%$ largest outcomes of X. Provides measure of distribution tail weight

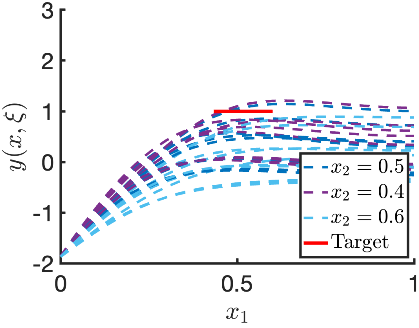


- Apply AVaR to $X = \frac{1}{2}\int_{D_o} \left(y(u;x,\boldsymbol{\xi}) - 1\right)_+^2 \,\mathrm{d}x$.
  Set $\beta = 0.95$, concerned with mitigating highest 5% of outcomes.

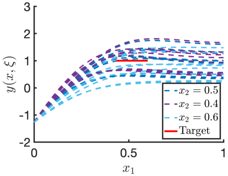
- Risk-averse AVaR optimization problem

$$\min_{u \in H^1(\partial D_c)} \ \mathsf{AVaR}_\beta \left[\frac{1}{2} \int_{D_o} \big(y(u; x, \boldsymbol{\xi}) - 1 \big)_+^2 \, \mathsf{d}x \right] + \frac{10^{-2}}{2} \|u\|^2_{H^1(\partial D_c)}.$$

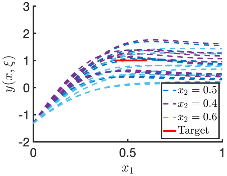
- Solve AVaR problem. Evaluate solution of state equation with AVaR control at same 10 samples of $\boldsymbol{\xi}$ used before.



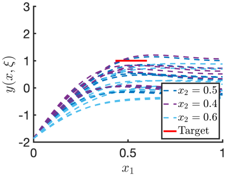
- Comparision



Deterministic



Risk-Neutral



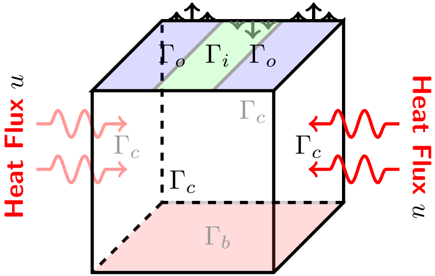
AVaR

- Incorporating uncertainties in parameters into computation of control matters.

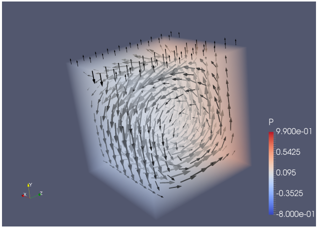
- Choice of problem formulation (risk-measure) application dependent.

Optimal Control of Boussinesq Flow Equations

Motivated by transport process in high pressure chemical vapor deposition (CVD) reactors



Uncontrolled



Optimal Control Problem

- Boussinesq flow equations on this domain are

$$
\begin{aligned}
-\mu\Delta v + (v\cdot\nabla)v + \nabla p + \eta T g &= 0, && \text{in } D,\\
\nabla\cdot v &= 0, && \text{in } D,\\
-\kappa\Delta T + v\cdot\nabla T &= 0, && \text{in } D,\\
v - v_i = 0, \qquad T &= 0, && \text{on } \Gamma_i,\\
v - v_o = 0, \qquad \kappa\nabla T\cdot n &= 0, && \text{on } \Gamma_o,\\
v = 0, \qquad T - T_b &= 0, && \text{on } \Gamma_b,\\
v = 0, \quad \kappa\nabla T\cdot n + h(u - T) &= 0, && \text{on } \Gamma_c,
\end{aligned}
$$

- Re is Reynolds number, Gr is Grashof number, Pr is Prandtl number

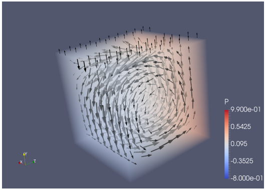
$$\mu = \frac{1}{\mathsf{Re}}, \quad \eta = \mu^2\mathsf{Gr}, \quad \text{and} \quad \kappa = \frac{\mu}{\mathsf{Pr}}.$$

- Re, Gr, and Pr random variables, uncertainty in h and T_b modeled using products of truncated KL. 53-dimensional random vector $\boldsymbol{\xi}$.
- Optimal Control Problem

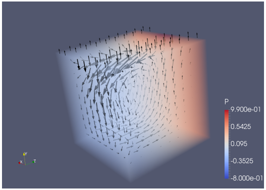
$$\min_{u\in\mathcal{U}} \mathcal{R}\left(\frac{1}{2}\int_D (\nabla\times v(u;\cdot,\boldsymbol{\xi}))(x)\ \mathsf{d}x\right) + \frac{\alpha}{2}\int_{\Gamma_c} |u(x)|^2\ \mathsf{d}x,$$

Optimal Control Results

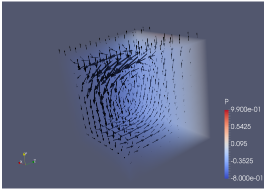
Uncontrolled



Risk Neutral



Risk Averse



- Entropic risk measure with $\sigma = 2$ used

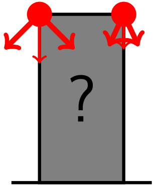
$$\mathcal{R}(X) = \sigma^{-1} \log \mathbb{E}[\exp(\sigma X)], \quad \sigma > 0,$$

- Risk-neutral and entropic risk significantly reduce the magnitude of the vorticity by approximately 2.3-fold.

Topology Optimization

- u material distribution. y displacement solve linear elasticity eqn.

$$\begin{aligned} -\nabla \cdot (\mathbf{E}(u) : \varepsilon) &= f, && \text{in } D, \\ \varepsilon &= \tfrac{1}{2}(\nabla y + \nabla y^\top), && \text{in } D, \\ \varepsilon n &= T, && \text{on } \Gamma_t, \\ y &= 0, && \text{on } \Gamma_d. \end{aligned}$$



- $(f = 0)$ Traction force ($\boldsymbol{\xi} = (\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)}, \boldsymbol{\xi}^{(3)}, \boldsymbol{\xi}^{(4)})$ uniformly distr.)

$$T(x, \xi) = \begin{cases} \left(\xi^{(1)} \cos(\xi^{(2)}), \xi^{(1)} \sin(\xi^{(2)})\right)^\top, & \text{if } x_1 \in [0, 1/8] \text{ and } x_2 = 2, \\ \left(\xi^{(3)} \cos(\xi^{(4)}), \xi^{(3)} \sin(\xi^{(4)})\right)^\top, & \text{if } x_1 \in [7/8, 1] \text{ and } x_2 = 2, \\ (0, 0)^\top, & \text{otherwise.} \end{cases}$$

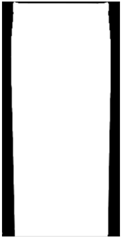
- Optimization problem

$$\min_{u \in L^2(D)} \int_D f(x) \cdot y(u; x)\, \mathsf{d}x + \int_{\Gamma_t} T(x) \cdot y(u; x)\, \mathsf{d}x$$

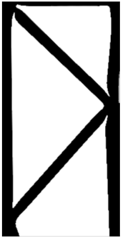
$$\text{subject to} \quad \int_D u(x)\, \mathsf{d}x \le v_0 |D|, \qquad 0 \le u \le 1 \ \text{ a.e.}$$

Topology Optimization Results

Deterministic



Risk Neutral



Risk Averse

- Intuitively, deterministic design places material in direction of the single deterministic load given by mean $\mathbb{E}[\boldsymbol{\xi}] = (1,\, 270^\circ,\, 1,\, 270^\circ)$ of uncertain parameters $\boldsymbol{\xi}$.
- Risk-neutral and risk-averse designs differ considerably from deterministic mean-value design, accounting for various loading scenarios modelled by traction load $T(\cdot, \boldsymbol{\xi})$.

Outline

Elliptic Model Problems

- Mathematical formulation of problem is foundation for reliable and efficient solution.
- Discuss statement of and well-posedness results for elliptic model problems.
- Begin with formal statement of two linear-quadratic elliptic optimal control problems.

Elliptic Model Problem - Deterministic

- Bounded domain $D \subset \mathbb{R}^n$ with boundary ∂D, given functions

$$f \in L^2(D),\ \kappa \in L^\infty(D),$$

such that

$$\kappa_{\max} \geq \kappa(x) \geq \kappa_{\min} > 0 \quad \text{a.e. in } D,$$

and given $u \in L^2(D)$, consider the elliptic PDE

$$-\nabla \cdot (\kappa(x) \nabla y(x)) = f(x) + u(x), \qquad x \in D, \qquad \text{(3a)}$$

$$y(x) = 0, \qquad x \in \partial D. \qquad \text{(3b)}$$

- Want to find $u \in L^2(D)$ such that solution $y(u; \cdot)$ of (3) is close to desired state $\widehat{y}$ a domain $D_o \subset D$ in the L^2 sense.
- Optimal control problem

$$\min_{u \in L^2(D)} \frac{1}{2} \int_{D_o} \big(y(u;x) - \widehat{y}(x)\big)^2 \, \mathsf{d}x + \frac{\alpha}{2} \int_D u(x)^2 \, \mathsf{d}x,$$

where $y(u; \cdot) \in H_0^1(D)$ is the solution of (3) given $u \in L^2(D)$.

Elliptic Model Problem - Random

- $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. State equation

$$-\nabla \cdot \big(\kappa(x,\omega)\nabla y(x,\omega)\big) = f(x,\omega) + u(x), \qquad x \in D, \omega \in \Omega, \quad \text{(4a)}$$
$$y(x,\omega) = 0, \qquad x \in \partial D, \omega \in \Omega, \quad \text{(4b)}$$

  where $u \in L^2(D)$, $f \in L^2_{\mathbb{P}}\big(\Omega, L^2(D)\big)$, $\kappa \in L^\infty_{\mathbb{P}}\big(\Omega, L^\infty(D)\big)$, and there exist measurable functions $\kappa_{\min}$ and $\kappa_{\max}$ such that

$$\kappa_{\max}(\omega) \geq \kappa(x,\omega) \geq \kappa_{\min}(\omega) > 0, \quad x \in D, \text{a.a. } \omega \in \Omega.$$

- Solution $\Omega \ni \omega \mapsto y(u;\omega;\cdot) \in H^1_0(D)$ of (4) is random field.
- Objective function

$$\Omega \ni \omega \mapsto \frac{1}{2}\int_{D_o} \big(y(u;\omega;x) - \widehat{y}(x)\big)^2 \,\mathrm{d}x + \frac{\alpha}{2}\int_D u(x)^2 \,\mathrm{d}x,$$

  is random variable.
- Risk neutral (expected value) optimal control problem

$$\min_{u \in L^2(D)} \int_\Omega \frac{1}{2}\int_{D_o} \big(y(u;\omega;x) - \widehat{y}(x)\big)^2 \,\mathrm{d}x\mathrm{d}\mathbb{P} + \frac{\alpha}{2}\int_D u(x)^2 \,\mathrm{d}x,$$

  where $y(u;\omega;\cdot) \in H^1_0(D)$ is the solution of (4) given $u \in L^2(D)$.

Elliptic Model Problem - Determinstic

- Well-posedness of elliptic model problem.
- Literature: Books by [Lions, 1971], [Hinze et al., 2009], [Tröltzsch, 2010], [Manzoni et al., 2021].

State Equation

- Hilbert space $\mathcal{U}$ is control space; Hilbert space $\mathcal{V}$ is state space.
- Given a control $u \in \mathcal{U}$, state $y \in \mathcal{V}$ satisfies variational equation

$$a(y, \varphi) + b(u, \varphi) = \ell(\varphi) \quad \forall \varphi \in \mathcal{V}, \tag{5}$$

  where
  - $a : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is a $\mathcal{V}$-coercive and continuous bilinear form, i.e., there exist $0 < a_{\min} \leq a_{\max}$ such that

$$a_{\min} \|y\|_{\mathcal{V}}^2 \leq a(y, y) \quad \text{and} \quad |a(y, \varphi)| \leq a_{\max} \|y\|_{\mathcal{V}} \|\varphi\|_{\mathcal{V}} \quad \forall y, \varphi \in \mathcal{V},$$

  - $b : \mathcal{U} \times \mathcal{V} \to \mathbb{R}$ is a continuous bilinear form, i.e., there exists $0 < b_{\max}$ such that

$$|b(u, \varphi)| \leq b_{\max} \|u\|_{\mathcal{U}} \|\varphi\|_{\mathcal{V}} \quad \forall u \in \mathcal{U}, \varphi \in \mathcal{V},$$

  - $\ell \in \mathcal{V}^*$ is a bounded linear form on $\mathcal{V}$.
- By Lax-Milgram Theorem: For every $u \in \mathcal{U}$, state equation (5) has a unique solution $y(u) \in \mathcal{V}$ with satisfies

$$\|y(u)\|_{\mathcal{V}} \leq a_{\min}^{-1} \big(\|\ell\|_{\mathcal{V}^*} + b_{\max} \|u\|_{\mathcal{U}} \big) \quad \forall u \in \mathcal{U}.$$

State Equation (cont.)

- Variational equation (5) equivalent to linear operator equation

$$Ay + Bu = \ell \qquad \text{in } \mathcal{V}^*,$$

where $A \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$ and $B \in \mathcal{L}(\mathcal{U}, \mathcal{V}^*)$ are defined by

$$\langle Ay, \varphi \rangle_{\mathcal{V}^*,\mathcal{V}} = a(y, \varphi), \quad \langle Bu, \varphi \rangle_{\mathcal{V}^*,\mathcal{V}} = b(u, \varphi) \quad \forall y, \varphi \in \mathcal{V}, u \in \mathcal{U}.$$

- Because a is $\mathcal{V}$-coercive, linear operator A is continuously invertible,

$$A^{-1} \in \mathcal{L}(\mathcal{V}^*, \mathcal{V}).$$

Optimal Control Problem

- Consider quadratic objective functional

$$\frac{1}{2}q(y,y) + c(y) + \frac{1}{2}r(u,u),$$

where

  - $q : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is symmetric, non-negative, continuous bilinear form, i.e., $q(y,\varphi) = q(\varphi,y)$ for all $y,\varphi \in \mathcal{V}$,

$$0 \le q(y,y) \quad \text{and} \quad |q(y,\varphi)| \le q_{\max} \|y\|_{\mathcal{V}} \|\varphi\|_{\mathcal{V}} \quad \forall y,\varphi \in \mathcal{V},$$

  - $r : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ is a symmetric, $\mathcal{U}$-coercive and continuous bilinear form, i.e., $r(u,\psi) = r(\psi,u)$ for all $u,\psi \in \mathcal{U}$, and

$$r_{\min} \|u\|_{\mathcal{U}}^2 \le r(u,u) \quad \text{and} \quad |r(u,\psi)| \le r_{\max} \|u\|_{\mathcal{U}} \|\psi\|_{\mathcal{U}} \quad \forall u,\psi \in \mathcal{U},$$

  - $c \in \mathcal{V}^*$ is a bounded linear form on $\mathcal{V}$.

- Optimal control problem

$$\min_{u \in \mathcal{U}} \ \frac{1}{2}q\big(y(u),y(u)\big) + c\big(y(u)\big) + \frac{1}{2}r(u,u),$$

where $y(u) \in \mathcal{V}$ is solution of state equation (5) given $u \in \mathcal{U}$.

Optimal Control Problem (Cont.)

- Optimal control problem

$$\min_{u \in \mathcal{U}} \ \frac{1}{2} q\big(y(u), y(u)\big) + c\big(y(u)\big) + \frac{1}{2} r(u, u),$$

  where $y(u) \in \mathcal{V}$ is solution of state equation (5) given $u \in \mathcal{U}$.
- Is a convex, elliptic, linear-quadratic optimal control problem.
- Objective function

$$f : \mathcal{U} \to \mathbb{R}, \quad u \mapsto f(u) := \frac{1}{2} q\big(y(u), y(u)\big) + c\big(y(u)\big) + \frac{1}{2} r(u, u),$$

  is Fréchet differentiable
- Fréchet derivative applied to δu is

$$f'(u)\delta u = r(u, \delta u) + b(\delta u, \lambda),$$

  where $\lambda \in \mathcal{V}$ solves adjoint equation

$$a(\varphi, \lambda) + q(u, \varphi) = -c(\varphi), \qquad \forall \varphi \in \mathcal{V}.$$

Optimal Control Problem - Optimality Conditions

- Optimal control problem

$$\min_{u \in \mathcal{U}} \ \frac{1}{2} q\big(y(u), y(u)\big) + c\big(y(u)\big) + \frac{1}{2} r(u, u),$$

  where $y(u) \in \mathcal{V}$ is solution of state equation (5) given $u \in \mathcal{U}$.
- Is a convex, elliptic, linear-quadratic optimal control problem.
- Has a unique solution $u \in \mathcal{U}$.
- $u \in \mathcal{U}$ solves optimal control problem if and only if there exist $y \in \mathcal{V}$ and $\lambda \in \mathcal{V}$ such that y, u, λ solve

$$\begin{aligned} a(\varphi, \lambda) + q(y, \varphi) &= -c(\varphi) && \forall \varphi \in \mathcal{V}, \\ r(u, \psi) + b(\psi, \lambda) &= 0 && \forall \psi \in \mathcal{U}, \\ a(y, \varphi) + b(u, \varphi) &= \ell(\varphi) && \forall \varphi \in \mathcal{V}. \end{aligned}$$

Example Optimal Control Problem - Optimality Conditions

- Optimal control problem

$$\min_{u \in L^2(D)} \; \frac{1}{2} \int_{D_o} \big(y(u;x) - \widehat{y}(x) \big)^2 \, \mathsf{d}x + \frac{\alpha}{2} \int_D u(x)^2 \, \mathsf{d}x,$$

where $y(u) \in H_0^1(D)$ is solution of state equation

$$\int_D \kappa(x) \nabla y(x) \nabla \varphi(x) \, \mathsf{d}x = \int_D \big(f(x) + u(x) \big) \varphi(x) \, \mathsf{d}x \quad \forall \varphi \in H_0^1(D).$$

- Has a unique solution $u \in L^2(D)$.
- $u \in L^2(D)$ solves optimal control problem if and only if there exist $y \in H_0^1(D)$ and $\lambda \in H_0^1(D)$ such that y, u, λ solve

$$\begin{aligned}
-\nabla \cdot \big(\kappa(x) \nabla \lambda(x) \big) &= -\big(y(x) - \widehat{y}(x) \big), & x &\in D, \\
\lambda(x) &= 0, & x &\in \partial D, \\
\alpha u(x) - \lambda(x) &= 0, & x &\in D, \\
-\nabla \cdot \big(\kappa(x) \nabla y(x) \big) &= f(x) + u(x), & x &\in D, \\
y(x) &= 0, & x &\in \partial D.
\end{aligned}$$

Elliptic Model Problem - Uncertainties in PDE

- Well-posedness of elliptic model problem.
- Conditions for almost all (a.a.) samples $\omega \in \Omega$ not enough for well-posedness, differentiabilty.
- Literature: [Heinkenschloss and Kouri, 2025].

Bochner Spaces

- $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, where
  - Ω is the set of outcomes,
  - $\mathcal{F} \subset 2^{\Omega}$ is σ-algebra of events, and
  - $\mathbb{P} : \mathcal{F} \to [0, 1]$ is a probability measure.
- Given Banach space $\mathcal{Y}$, we define the Bochner spaces

$$L^q_{\mathbb{P}}(\Omega, \mathcal{Y}) := \Big\{ v : \Omega \to \mathcal{Y} \;\Big|\; v \text{ is strongly measurable and} \int_{\Omega} \|y(\omega)\|^q_{\mathcal{Y}} \, \mathrm{d}\mathbb{P}(\omega) < \infty \Big\},$$

for $q \in [1, \infty)$, and

$$L^{\infty}_{\mathbb{P}}(\Omega, \mathcal{Y}) := \Big\{ v : \Omega \to \mathcal{Y} \;\Big|\; v \text{ is strongly measurable and } \mathbb{P} - \operatorname*{ess\,sup}_{\omega \in \Omega} \|y(\omega)\|_{\mathcal{Y}} < \infty \Big\}.$$

When $\mathcal{Y} = \mathbb{R}$, we simplify notation to $L^p_{\mathbb{P}}(\Omega, \mathbb{R}) = L^p_{\mathbb{P}}(\Omega)$.

State Equation

- Hilbert space $\mathcal{U}$ is control space;
  separable Hilbert space $\mathcal{V}$ related to state space.
- $(\Omega, \mathcal{F}, \mathbb{P})$ complete probability space; $\mathcal{B}$ Borel σ-algebra on $\mathbb{R}$.
- Given a control $u \in \mathcal{U}$ and realization $\omega \in \Omega$, consider

$$a(y, \varphi, \omega) + b(u, \varphi, \omega) = \ell(\varphi, \omega) \quad \forall \varphi \in \mathcal{V}, \tag{6}$$

  Assume (pointwise a.a. $\omega \in \Omega$ assumptions)
  - $a(\cdot, \cdot, \omega) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}, \quad b(\cdot, \cdot, \omega) : \mathcal{U} \times \mathcal{V} \to \mathbb{R}, \quad \ell(\cdot, \omega) \in \mathcal{V}^*.$
  - For each $y, \varphi \in \mathcal{V}$, $u \in \mathcal{U}$, the functions

$$a(y, \varphi, \cdot), b(u, \varphi, \cdot) : \Omega \to \mathbb{R} \text{ are } (\mathcal{F}, \mathcal{B}) \text{ measurable.}$$

  - There exist measurable functions $0 < a_{\min}(\omega) \le a_{\max}(\omega)$ and $0 < b_{\max}(\omega)$ such that for a.a. $\omega \in \Omega$,

$$\begin{aligned}
a(y, y, \omega) &\ge a_{\min}(\omega) \|y\|_{\mathcal{V}}^2 && \forall y \in \mathcal{V}, \\
|a(y, \varphi, \omega)| &\le a_{\max}(\omega) \, \|y\|_{\mathcal{V}} \|\varphi\|_{\mathcal{V}} && \forall y, \varphi \in \mathcal{V}, \\
|b(u, \varphi, \omega)| &\le b_{\max}(\omega) \, \|u\|_{\mathcal{U}} \|\varphi\|_{\mathcal{V}} && \forall u \in \mathcal{U}, \varphi \in \mathcal{V}.
\end{aligned}$$

- $a : \mathcal{V} \times \mathcal{V} \times \Omega \to \mathbb{R}$ is a Carathéodory function;
  for every $u \in \mathcal{U}$, $b(u, \cdot, \cdot) : \mathcal{V} \times \Omega \to \mathbb{R}$ is a Carathéodory function.
  Composition of a, b with Lebesque meas. function is Lebesque meas.

State Equation (cont.)

- For every control $u \in \mathcal{U}$ and a.a. realizations $\omega \in \Omega$ the parameterized state equation

$$a(y, \varphi, \omega) + b(u, \varphi, \omega) = \ell(\varphi, \omega) \quad \forall \varphi \in \mathcal{V},$$

  has a unique solution $y(u; \omega) \in \mathcal{V}$.

- However, need measurability and integrability properties of the function $y(u; \cdot) : \Omega \to \mathcal{V}$.

- Two possible routes:
  1. Formulate state equation as variational equation in $L^2_{\mathbb{P}}\big(\Omega, \mathcal{V}\big)$.
  2. Prove that $\omega \mapsto y(u, \omega)$ is measurable by showing it is the limit of measurable functions. Then use properties of $a_{\min}(\omega), a_{\max}(\omega), b_{\max}(\omega), l(\omega)$ to establish integrability result of $\omega \mapsto y(u, \omega)$.

1. State Equation in $L^2_{\mathbb{P}}(\Omega, \mathcal{V})$.

- Use variational theory (Lax Milgram) in Hilbert space $L^2_{\mathbb{P}}(\Omega, \mathcal{V})$.
- For every control $u \in \mathcal{U}$ state equation

$$\int_\Omega a(y(\omega), \varphi(\omega), \omega)\, \mathsf{d}\mathbb{P}(\omega) + \int_\Omega b(u, \varphi(\omega), \omega)\, \mathsf{d}\mathbb{P}(\omega)$$
$$= \int_\Omega \ell(\varphi(\omega), \omega)\, \mathsf{d}\mathbb{P}(\omega), \quad \forall \varphi \in L^2_{\mathbb{P}}(\Omega, \mathcal{V}).$$

- Assume

$$a(y, y, \omega) \geq a_{\min}(\omega) \|y\|^2_{\mathcal{V}} \qquad \forall y \in \mathcal{V},$$

with

$$\operatorname*{ess\,inf}_{\omega \in \Omega} a_{\min}(\omega) > 0,$$

and $a_{\max} \in L^\infty_{\mathbb{P}}(\Omega)$, $\quad b_{\max} \in L^2_{\mathbb{P}}(\Omega)$, $\quad \ell \in L^2_{\mathbb{P}}(\Omega, \mathcal{V}^*)$.

1. State Equation in $L^2_{\mathbb{P}}(\Omega, \mathcal{V})$ (cont.).

- State equation

$$\int_\Omega a(y(\omega), \varphi(\omega), \omega)\, \mathrm{d}\mathbb{P}(\omega) + \int_\Omega b(u, \varphi(\omega), \omega)\, \mathrm{d}\mathbb{P}(\omega)$$
$$= \int_\Omega \ell(\varphi(\omega), \omega)\, \mathrm{d}\mathbb{P}(\omega), \quad \forall \varphi \in L^2_{\mathbb{P}}(\Omega, \mathcal{V}).$$

- Use variational theory (Lax Milgram) in Hilbert space $L^2_{\mathbb{P}}(\Omega, \mathcal{V})$: $\forall u \in \mathcal{U}$, state equation has a unique solution $y(u) \in L^2_{\mathbb{P}}(\Omega, \mathcal{V})$, and

$$\|y(u)\|_{L^2_{\mathbb{P}}(\Omega,\mathcal{V})} \le \frac{1}{\text{ess inf}_{\omega\in\Omega}\, a_{\min}} \big(\|\ell\|_{L^2_{\mathbb{P}}(\Omega,\mathcal{V}^*)} + \|b_{\max}\|_{L^\infty_{\mathbb{P}}(\Omega)} \|u\|_{\mathcal{U}} \big).$$

1. State Equation in $L^2_{\mathbb{P}}(\Omega, \mathcal{V})$ (cont.).

- Apply to

$$
\begin{aligned}
-\nabla \cdot \big(\kappa(x,\omega)\nabla y(x,\omega)\big) &= f(x,\omega) + u(x), \qquad && x \in D, \omega \in \Omega, \\
y(x,\omega) &= 0, && x \in \partial D, \omega \in \Omega,
\end{aligned}
$$

where $\kappa \in L^\infty_{\mathbb{P}}\big(\Omega, L^\infty(D)\big)$ such that

$$
\kappa_{\max}(\omega) \ge \kappa(x,\omega) \ge \kappa_{\min}(\omega) > 0, \quad x \in D, \text{a.a. } \omega \in \Omega.
$$

- For lognormally distributed

$$
\kappa(x,\omega) = \exp\left(\sum_{m=1}^{M} b^{(m)}(x)\, \boldsymbol{\xi}^{(m)}(\omega) \right),
$$

$b^{(m)} \in L^\infty(D)$, $\boldsymbol{\xi}^{(m)} \sim N(0,1)$, $m = 1, \ldots, M$, i.i.d.,
κ not uniformly bded away from zero $\Rightarrow$ variational theory doesn't apply!

- Can select the lower bound

$$
\kappa_{\min}(\omega) = \exp\left(-\sum_{m=1}^{M} \|b^{(m)}\|_{L^\infty(D)} \, |\boldsymbol{\xi}^{(m)}(\omega)| \right),
$$

Satisfies $\kappa_{\min}^{-1} \in L^{kq}_{\mathbb{P}}(\Omega)$ $\forall k \in \mathbb{N}$, $1 < q < \infty$. [Babuška et al., 2007]

2. State Equation in $L^2_{\mathbb{P}}(\Omega, \mathcal{V})$ (cont.).

- $\omega \mapsto y(u,\omega)$ is the limit of the solutions $y_n \in L^2_{\mathbb{P}}(\Omega, \mathcal{V})$ of

$$a(y,\varphi,\omega) + \frac{1}{n}\langle y, \varphi\rangle_{\mathcal{V}} + b(u,\varphi,\omega) = \ell(\varphi,\omega) \quad \forall \varphi \in \mathcal{V},$$

- Use properties of $a_{\min}(\omega), a_{\max}(\omega), b_{\max}(\omega), l(\omega)$ to establish integrability result of $\omega \mapsto y(u,\omega)$.
- **Existence and Uniqueness:** Let $p, q \geq 1$ with $1/p + 1/q = 1$ and $k \in \mathbb{N}$. If $\ell \in L^{kp}_{\mathbb{P}}(\Omega, \mathcal{V}^*)$, $b_{\max} \in L^{kp}_{\mathbb{P}}(\Omega)$, and $a_{\min}^{-1} \in L^{kq}_{\mathbb{P}}(\Omega)$, then unique solution $y(u;\cdot)$ of (6) satisfies $y(u;\cdot) \in L^k_{\mathbb{P}}(\Omega, \mathcal{V})$ and

$$\begin{aligned}\int_\Omega \|y(u;\omega)\|^k_{\mathcal{V}}\, d\mathbb{P}(\omega) \leq & \Big(\int_\Omega \big(\frac{1}{a_{\min}(\omega)}\big)^{kq}\, d\mathbb{P}(\omega)\Big)^{1/q} \\ & \times \Big(\int_\Omega \big(\|\ell(\omega)\|_{\mathcal{V}^*} + b_{\max}(\omega)\|u\|_{\mathcal{U}}\big)^{kp}\, d\mathbb{P}(\omega)\Big)^{1/p}.\end{aligned}$$

- Special case of variational result if one chooses $p = 1$, $q = \infty$, $k = 2$.
- Applies to problem with lognormally distributed coeff. If, for some $\epsilon > 0$, $f \in L^{k(1+\epsilon)}_{\mathbb{P}}(\Omega, L^2(D))$, solution $y(u;\cdot) \in L^k_{\mathbb{P}}(\Omega, H^1_0(\Omega))$.

Optimal Control Problem

- We consider a quadratic fiunctionl

$$\frac{1}{2}q(y,y,\omega) + c(y;\omega) + \frac{1}{2}r(u,u),$$

  where
  - For $\omega \in \Omega$, $q(\cdot,\cdot,\omega) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is symmetric, non-negative, continuous bilinear form, i.e., $q(y,\varphi,\omega) = q(\varphi,y,\omega)$ for all $y,\varphi \in \mathcal{V}$,

$$0 \le q(y,y,\omega), \quad \text{and}$$
$$|q(y,\varphi,\omega)| \le q_{\max}(\omega)\,\|y\|_{\mathcal{V}}\|\varphi\|_{\mathcal{V}} \quad \forall y,\varphi \in \mathcal{V}, \qquad q_{\max} \in L^{\infty}_{\mathbb{P}}(\Omega).$$

  - (as before) $r : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ is a symmetric, $\mathcal{U}$-coercive and continuous bilinear form, i.e., $r(u,\psi) = r(\psi,u)$ for all $u,\psi \in \mathcal{U}$, and

$$r_{\min}\|u\|^2_{\mathcal{U}} \le r(u,u) \quad \text{and} \quad |r(u,\psi)| \le r_{\max}\,\|u\|_{\mathcal{U}}\|\psi\|_{\mathcal{U}} \quad \forall u,\psi \in \mathcal{U},$$

  - $c \in L^2_{\mathbb{P}}\big(\Omega, \mathcal{V}^*\big)$.
- If state $y(u;\cdot) \in L^2_{\mathbb{P}}\big(\Omega, \mathcal{V}\big)$, then

$$\omega \mapsto \frac{1}{2}q\big(y(u;\omega), y(u;\omega), \omega\big) + c\big(y(u;\omega), \omega\big) + \frac{1}{2}r(u,u)$$

  is integrable

Optimal Control Problem (Cont.)

- Risk-neutral optimal control problem: If state $y(u;\cdot) \in L^2_{\mathbb{P}}\big(\Omega, \mathcal{V}\big)$, can minimize expected cost

$$\min_{u \in \mathcal{U}} \int_\Omega \frac{1}{2} q\big(y(u;\omega), y(u;\omega), \omega\big) + c\big(y(u;\omega), \omega\big) \, \mathrm{d}\mathbb{P}(\omega) + \frac{1}{2} r(u,u),$$

- Control $u \in \mathcal{U}$ is deterministic because we need to decide on control before uncertainty is revealed, i.e., before outcome $\omega \in \Omega$ is known.

- If previous existence result holds with $k = 2$, risk-neutral optimal control problem is convex, has a unique solution $u \in \mathcal{U}$.

Three Maps Associated with Objective

- Control-to-state map

$$
\begin{aligned}
S : \mathcal{U} &\to L^2_{\mathbb{P}}\big(\Omega, \mathcal{V}\big), \\
u &\mapsto y(u; \cdot) \qquad \big(y(u;\omega) \text{ solves (6)}\big).
\end{aligned}
$$

- Pointwise

$$
\begin{aligned}
J : \mathcal{V} \times \Omega &\to \mathbb{R}, \\
(y, \omega) &\mapsto \frac{1}{2} q\big(y, y, \omega\big) + c\big(y, \omega\big),
\end{aligned}
$$

  and associated map

$$
\begin{aligned}
\mathcal{J} : L^2_{\mathbb{P}}\big(\Omega, \mathcal{V}\big) &\to L^1_{\mathbb{P}}\big(\Omega\big), \\
y &\mapsto [\mathcal{J}(y)](\cdot) = J(y(\cdot), \cdot).
\end{aligned}
$$

- Expected value (risk measure)

$$
\begin{aligned}
\mathcal{R} : L^1_{\mathbb{P}}\big(\Omega\big) &\to \mathbb{R} \\
\boldsymbol{\xi} &\mapsto \mathbb{E}[\boldsymbol{\xi}] = \int_\Omega \boldsymbol{\xi}(\omega)\, \mathrm{d}\mathbb{P}(\omega).
\end{aligned}
$$

- Optimal control problem $\min_{u \in \mathcal{U}} \; \mathcal{R}\big[\mathcal{J}\big(S(u)\big)\big] + \frac{1}{2} r(u, u)$.

Fréchet Differentiability

- Control-to-state map:
  - If 'pointwise assumptions' hold, $\mathcal{U} \ni u \mapsto y(u;\omega) \in \mathcal{V}$ is Fréchet diff'bel. a.a. $\omega \in \Omega$.
  - If 'existence & uniquess assumptions' hold with $k = 2$, $S : \mathcal{U} \to L^2_{\mathbb{P}}\big(\Omega, \mathcal{V}\big)$ is Fréchet diff'bel., Fréchet derivative can be computed pointwise for a.a. $\omega \in \Omega$.

- Objective function maps
  - If 'pointwise assumptions' on q, c hold, $J : \mathcal{V} \times \Omega \to \mathbb{R}$, is Fréchet diff'bel. in y on $\mathcal{V} \times \Omega$.
  - If integrability conditions on q_{max} hold, $\mathcal{J} : L^2_{\mathbb{P}}\big(\Omega, \mathcal{V}\big) \to L^1_{\mathbb{P}}\big(\Omega\big)$ is Fréchet diff'bel.

- Composition: If all above condition hold, $\mathcal{J} \circ S$ is Fréchet differentiable.

Fréchet Differentiability (cont.)

- If all above conditions hold, $\mathcal{J} \circ S$ is Fréchet differentiable and the Fréchet derivative is

$$\begin{aligned}[\mathcal{J}_y(S(u))\, S'(u)\delta u](\cdot) &= q\big(y(u;\cdot), y_u(u;\cdot)\delta u, \cdot\big) + c\big(y_u(u;\cdot)\delta u, \cdot\big) \\ &= b(\delta u, \lambda(\cdot), \cdot),\end{aligned}$$

  where $\lambda \in L^1_{\mathbb{P}}\big(\Omega, \mathcal{V}\big)$ is solution of (pointwise) adjoint equation

$$a(\varphi, \lambda(\omega), \omega) + q(y(u;\omega), \varphi, \omega) = -c(\varphi, \omega), \quad \forall \varphi \in \mathcal{V}, \text{ a.a. } \omega \in \Omega.$$

  If, in addition, $a_{\min}^{-1} \in L^\infty_{\mathbb{P}}\big(\Omega\big)$, then $\lambda \in L^2_{\mathbb{P}}\big(\Omega, \mathcal{V}\big)$.

- If all above conditions hold, $f = \mathbb{E} \circ \mathcal{J} \circ S$ is Fréchet differentiable; Fréchet derivative applied to δu is

$$f'(u)\delta u = \int_\Omega b(\delta u, \lambda(\omega), \omega),\, \mathrm{d}\mathbb{P}(\omega) + r(u, \delta u).$$

Optimal Control Problem - Optimality Conditions

- Risk-neutral optimal control problem: If state $y(u;\cdot) \in L^2_{\mathbb{P}}\big(\Omega, \mathcal{V}\big)$, can minimize expected cost

$$\min_{u \in \mathcal{U}} \int_\Omega \frac{1}{2} q\big(y(u;\omega), y(u;\omega), \omega\big) + c\big(y(u;\omega), \omega\big)\, \mathsf{d}\mathbb{P}(\omega) + \frac{1}{2} r(u,u).$$

- If state equation existence result holds with $k=2$, risk-neutral optimal control problem Is convex. Has a unique solution $u \in \mathcal{U}$.

- $u \in \mathcal{U}$ solves risk-neutral optimal control problem if and only if there exist $y \in L^2_{\mathbb{P}}\big(\Omega, \mathcal{V}\big)$ and $\lambda \in L^1_{\mathbb{P}}\big(\Omega, \mathcal{V}\big)$ such that y, u, λ solve

$$\begin{aligned} a(\varphi, \lambda(\omega), \omega) + q(y(\omega), \varphi, \omega) &= -c(\varphi, \omega) && \forall \varphi \in \mathcal{V},\ \text{a.a. } \omega \in \Omega, \\ \int_\Omega b(\psi, \lambda(\omega), \omega)\, \mathsf{d}\mathbb{P}(\omega) + r(u, \psi) &= 0 && \forall \psi \in \mathcal{U}, \\ a(y(\omega), \varphi, \omega) + b(u, \varphi, \omega) &= \ell(\varphi, \omega) && \forall \varphi \in \mathcal{V},\ \text{a.a. } \omega \in \Omega. \end{aligned}$$

Example Optimal Control Problem - Optimality Conditions

- $\eta \in L^\infty_{\mathbb{P}}\big(\Omega, L^\infty(D)\big)$, $\eta \geq 0$ a.e. in $D \times \Omega$. Optimal control problem

$$\min_{u \in L^2(D)} \int_\Omega \frac{1}{2} \int_D \eta(x,\omega)\big(y(u;x,\omega) - \widehat{y}(x)\big)^2 \,\mathrm{d}x\,\mathrm{d}\mathbb{P}(\omega) + \frac{\alpha}{2} \int_D u(x)^2 \,\mathrm{d}x,$$

where for a.a. $\omega \in \Omega$, $y(u,\cdot,\omega) \in H^1_0(D)$ solves

$$\int_D \kappa(x,\omega) \nabla y(x,\omega) \nabla \varphi(x)\,\mathrm{d}x = \int_D \big(f(x,\omega) + u(x)\big)\varphi(x)\,\mathrm{d}x \;\forall \varphi \in H^1_0(D).$$

- $u \in L^2(D)$ solves optimal control problem if and only if there exist $y \in L^2_{\mathbb{P}}\big(\Omega, H^1_0(D)\big)$, $\lambda \in L^1_{\mathbb{P}}\big(\Omega, H^1_0(D)\big)$ such that

$$-\nabla\cdot\big(\kappa(x,\omega)\nabla\lambda(x,\omega)\big) = -\eta(x,\omega)\big(y(x,\omega) - \widehat{y}(x)\big), \; x \in D, \text{a.a. } \omega \in \Omega,$$

$$\begin{aligned}
\lambda(x,\omega) &= 0, & x \in \partial D, \text{a.a. } \omega \in \Omega,\\
\int_\Omega \lambda(x,\omega)\,\mathrm{d}\mathbb{P}(\omega) &= \alpha u(x), & x \in D,\\
-\nabla\cdot\big(\kappa(x,\omega)\nabla y(x,\omega)\big) &= f(x,\omega) + u(x), & x \in D, \text{a.a. } \omega \in \Omega,\\
y(x,\omega) &= 0, & x \in \partial D, \text{a.a. } \omega \in \Omega.
\end{aligned}$$

- Control deterministic $\Rightarrow$ coupling $\int_\Omega \lambda(x,\omega)\,\mathrm{d}\mathbb{P}(\omega) = \alpha u(x)$.

Note on Finite Noise Assumption

- So far used $(\Omega, \mathcal{F}, \mathbb{P})$ complete probability space.
- Often consider a vector $\boldsymbol{\xi} : (\Omega, \mathcal{F}) \to (\Xi, \mathcal{B}_\Xi)$ of random variables. $\Xi \subset \mathbb{R}^M$ is a nonempty set and $\mathcal{B}_\Xi$ is the Borel σ-algebra on Ξ.
- All our examples depended on finitely many random variables.
- Finite noise assumption needed for some methods, e.g., sparse grid discretization and quasi-Monte Carlo.
- Sometimes problem can be well-approximated by a finite noise problem, e.g., approximate diffusion coefficient

$$
\begin{aligned}
\kappa(x,\omega) &= \exp\left(\sum_{m=1}^{\infty} b^{(m)}(x)\, \boldsymbol{\xi}^{(m)}(\omega) \right) \\
&\approx \kappa_M(x, \boldsymbol{\xi}) = \exp\left(\sum_{m=1}^{M} b^{(m)}(x)\, \boldsymbol{\xi}^{(m)} \right).
\end{aligned}
$$

- Distribution (or law) of $\boldsymbol{\xi}$ on σ-algebra $\sigma(\boldsymbol{\xi}) = \{\boldsymbol{\xi}^{-1}(B) \mid B \in \mathcal{B}_\Xi\}$ is $\mathbb{P}^\xi = \mathbb{P} \circ \boldsymbol{\xi}^{-1}$.
- All previous results apply with $(\Omega, \mathcal{F}, \mathbb{P}) \to (\Xi, \mathcal{B}_\Xi, \mathbb{P}^\xi)$, $\omega \to \xi$.

Elliptic Model Problem - Discretization

- Numerical solution requires discretization of the problem.
- Crucial component in solution process.
- Discretization of optimal control problem typically builds on discretization of underlying PDE.
- Good discretization scheme for underlying PDE is necessary but not sufficient for good discretization scheme of optimal control problem.

Elliptic Model Problem - Determinstic

- Example, conforming Galerkin finite-element approximation:

$$\mathcal{U}_h = \text{span}\{\psi_1, \dots, \psi_{n_u}\} \subset \mathcal{U}, \quad \mathcal{V}_h = \text{span}\{\varphi_1, \dots, \varphi_{n_y}\} \subset \mathcal{V}.$$

- Approximate control and state by

$$u_h = \sum_{i=1}^{n_u} \mathbf{u}_i \psi_i, \quad y_h = \sum_{i=1}^{n_y} \mathbf{y}_i \varphi_i.$$

- Replace weak form of state equation by

$$a(y_h, \varphi) + b(u_h, \varphi) = \ell(\varphi), \quad \forall \varphi \in \mathcal{V}_h.$$

- This leads to

$$\mathbf{A}\mathbf{y} + \mathbf{B}\mathbf{u} = \mathbf{b},$$

with

$$\begin{aligned} \mathbf{A}_{ij} &= a(\varphi_j, \varphi_i), & i, j = 1, \dots n_y, \\ \mathbf{B}_{ij} &= b(\psi_j, \varphi_i), & i = 1, \dots n_y, j = 1, \dots n_u, \\ \mathbf{b}_i &= \ell(\varphi_i), & i = 1, \dots n_y. \end{aligned}$$

- Coercivity of a implies invertibility of $\mathbf{A} \in \mathbb{R}^{n_y \times n_y}$.

$$\mathbf{y}(\mathbf{u}) = \mathbf{A}^{-1}(\mathbf{b} - \mathbf{B}\mathbf{u}).$$

- Insert state and control approximation int objective to get

$$\frac{1}{2}\mathbf{y}^\top \mathbf{Q}\,\mathbf{y} + \mathbf{c}^\top \mathbf{y} + \frac{1}{2}\mathbf{u}^\top \mathbf{R}\,\mathbf{u},$$

- $\mathbf{Q} = \mathbf{Q}^\top \in \mathbb{R}^{n_y \times n_y}$, $\mathbf{R} = \mathbf{R}^\top \in \mathbb{R}^{n_u \times n_u}$, $\mathbf{c} \in \mathbb{R}^{n_y}$,

  $$\mathbf{Q}_{ij} = q(\varphi_i, \varphi_j),\ i,j = 1, \ldots n_y, \quad \mathbf{R}_{ij} = r(\psi_i, \psi_j),\ i,j = 1, \ldots n_u,$$

  and $\mathbf{c}_i = c(\varphi_i)$, $i = 1, \ldots n_y$.
- Assumptions on bilinear forms q and r imply that $\mathbf{Q} \in \mathbb{R}^{n_y \times n_y}$ is symmetric positive semidefinite and $\mathbf{R} \in \mathbb{R}^{n_u \times n_u}$ is symmetric positive definite.
- Discretization optimal control problem

$$\begin{aligned}\min_{\mathbf{u} \in \mathbb{R}^{n_u}}\ & \frac{1}{2}\mathbf{u}^\top \big(\mathbf{B}^\top \mathbf{A}^{-\top} \mathbf{Q}\,\mathbf{A}^{-1}\mathbf{B} + \mathbf{R}\big)\,\mathbf{u} - \Big(\mathbf{B}^\top \mathbf{A}^{-\top}\big(\mathbf{c} + \mathbf{Q}\,\mathbf{A}^{-1}\mathbf{b}\big)\Big)^\top \mathbf{u} \\ & + \frac{1}{2}\mathbf{b}^\top \mathbf{A}^{-\top} \mathbf{Q}\,\mathbf{A}^{-1}\mathbf{b} + \mathbf{c}^\top \mathbf{A}^{-1}\mathbf{b}.\end{aligned}$$

- Discretization optimal control problem

$$\min_{\mathbf{u}\in\mathbb{R}^{n_u}} \frac{1}{2}\mathbf{u}^\top\big(\mathbf{B}^\top\mathbf{A}^{-\top}\mathbf{Q}\,\mathbf{A}^{-1}\mathbf{B}+\mathbf{R}\big)\,\mathbf{u} - \big(\mathbf{B}^\top\mathbf{A}^{-\top}\big(\mathbf{c}+\mathbf{Q}\,\mathbf{A}^{-1}\mathbf{b}\big)\big)^\top\mathbf{u} + \frac{1}{2}\mathbf{b}^\top\mathbf{A}^{-\top}\mathbf{Q}\,\mathbf{A}^{-1}\mathbf{b} + \mathbf{c}^\top\mathbf{A}^{-1}\mathbf{b}.$$

- Has a unique solution $\mathbf{u}\in\mathbb{R}^{n_u}$.
- $\mathbf{u}\in\mathbb{R}^{n_u}$ solves Discretization optimal control problem if and only if

$$\big(\mathbf{B}^\top\mathbf{A}^{-\top}\mathbf{Q}\,\mathbf{A}^{-1}\mathbf{B}+\mathbf{R}\big)\,\mathbf{u} = \mathbf{B}^\top\mathbf{A}^{-\top}\big(\mathbf{c}+\mathbf{Q}\,\mathbf{A}^{-1}\mathbf{b}\big),$$

  and if and only if there exists $\mathbf{y}\in\mathbb{R}^{n_y}$ and $\boldsymbol{\lambda}\in\mathbb{R}^{n_y}$ such that

$$\begin{pmatrix} \mathbf{Q} & \mathbf{0} & \mathbf{A}^\top \\ \mathbf{0} & \mathbf{R} & \mathbf{B}^\top \\ \mathbf{A} & \mathbf{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ \mathbf{u} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} -\mathbf{c} \\ \mathbf{0} \\ \mathbf{b} \end{pmatrix}.$$

- Many solvers for such systems. See, e.g., [Benzi et al., 2005], [Borzì and Schulz, 2009], [Borzì and Schulz, 2012], [Pearson and Pestana, 2020], [Wathen, 2015].

Basic Discretization Error

- Problem minimize$_{u\in\mathcal{U}} f(u) = \widetilde{f}(y(u), u)$.
- Approximate problem (e.g., discretization), $\mathcal{U}_h \subset \mathcal{U}$

$$\text{minimize}_{u_h\in\mathcal{U}_h} f_h(u_h) = \widetilde{f}_h(y_h(u_h), u_h).$$

- u^* and u_h^* solutions. Taylor expansion

$$\nabla f(u^*) - \nabla f(u_h^*) = \nabla^2 f\big(u_h^* + t(u^* - u_h^*)\big)(u^* - u_h^*).$$

  (Local) convexity of f (true for quadratic model problem)

$$\begin{aligned}\langle \nabla f(u^*) - \nabla f(u_h^*), u^* - u_h^* \rangle_{\mathcal{U}} &= \langle \nabla^2 f\big(u_h^* + t(u^* - u_h^*)\big)(u^* - u_h^*), u^* - u_h^* \rangle_{\mathcal{U}} \\ &\geq \sigma \|u^* - u_h^*\|_{\mathcal{U}}^2.\end{aligned}$$

  Hence

$$\sigma\|u^* - u_h^*\|_{\mathcal{U}} \leq \|\underbrace{\nabla f(u^*)}_{=0} - \nabla f(u_h^*)\|_{\mathcal{U}} = \|\underbrace{\nabla f_h(u_h^*)}_{=0} - \nabla f(u_h^*)\|_{\mathcal{U}}.$$

- Error between solutions u^*, u_h^* bounded by error in gradients at u_h^*.
- Not sufficient to approximate state $y(u)$, i.e., function $f(u)$.
  Must also approximate $\nabla f(u)$, i.e., the adjoint $\lambda(u)$.

- In elliptic model problem, must interpret

$$\mathbf{A}^\top \boldsymbol{\lambda} + \mathbf{Q}\mathbf{y} = -\mathbf{c}$$

as discretization of adjoint equation

$$a(\varphi, \lambda) + q(y, \varphi) = -c(\varphi) \qquad \forall \varphi \in \mathcal{V},$$

- In conforming Galerkin disc., $\mathbf{A}^\top \boldsymbol{\lambda} + \mathbf{Q}\mathbf{y} = -\mathbf{c}$ is equivalent to

$$a(\varphi, \lambda_h) + q(y_h, \varphi) = -c(\varphi), \quad \forall \varphi \in \mathcal{V}_h,$$

where

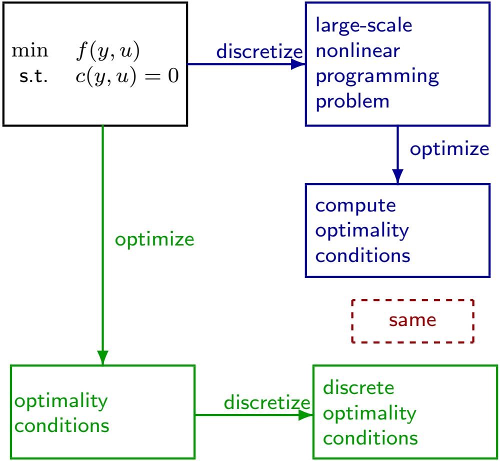
$$\lambda_h = \sum_{i=1}^{n_y} \boldsymbol{\lambda}_i \varphi_i \in \mathcal{V}_h.$$

- Can prove discretization error result of the form

$$\|u^* - u_h^*\|_{L^2(D)} + h\|y(u^*) - y_h(u_h^*)\|_{H_0^1(D)} \le ch^2 \big(\|y(u^*)\|_{L^2(D)} + \|u^*\|_{L^2(D)}\big).$$

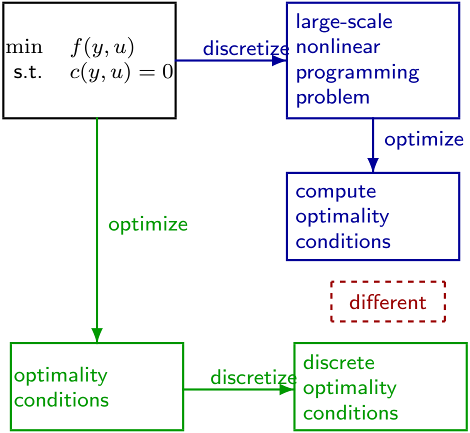
[Hinze et al., 2009, Theorem 3.5]

Discretize-then-optimize = optimize-then-discretize



Not Always this Nice and Clean

- For stabilized finite element methods [Collis and Heinkenschloss, 2002], [Abraham et al., 2004], [Heinkenschloss and Leykekhman, 2010], [Quarteroni, 2009, §16.13], and
- several discontinuous Galerkin methods [Leykekhman, 2012] and [Leykekhman and Heinkenschloss, 2012].



SUPG Discretization of Advection Diffusion OCP

$$\min \frac{1}{2}\int_\Omega (y(x)-\widehat{y}(x))^2 dx + \frac{\alpha}{2}\int_\Omega u^2(x)dx$$

subject to

$$\begin{aligned} -\epsilon\Delta y(x) + c(x)\cdot\nabla y(x) + r(x)y(x) &= f(x)+u(x), \quad && x\in\Omega,\\ y(x) &= d(x), && x\in\Gamma_d,\\ \epsilon\frac{\partial}{\partial n}y(x) &= g(x), && x\in\Gamma_n, \end{aligned}$$

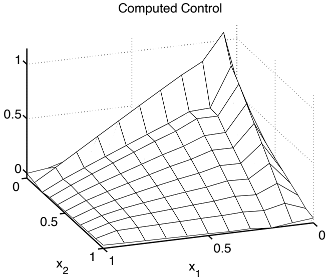
where $f,\widehat{y}\in L^2(\Omega)$, $\alpha>0$,

$$\epsilon>0, \quad c\in\left(W^{1,\infty}(\Omega)\right)^2, \quad r\in L^\infty(\Omega),$$

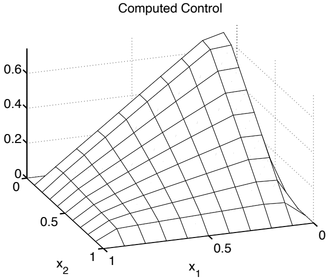
$$r(x)-\tfrac{1}{2}\nabla\cdot c(x)\ge r_0>0 \text{ a.e. in } \Omega, \quad n\cdot c(x)\ge 0 \text{ on } \Gamma_n.$$

We are interested in the case $\epsilon \ll \|c(x)\|$.

Linear Finite Elements

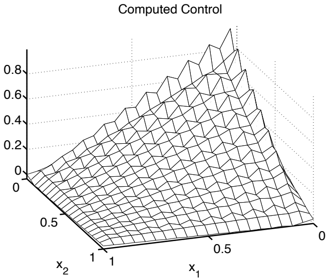


discretize-then-optimize

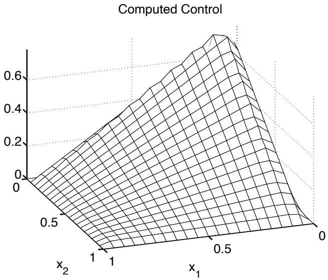


optimize-then-discretize

Quadratic Finite Elements



discretize-then-optimize



optimize-then-discretize

Galerkin/Least Squares (GaLS) Stabilized FEM for Oseen

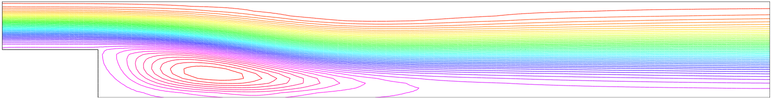
$$\min \frac{1}{2}\int_{\Omega_{\text{obs}}} |\nabla\times\mathbf{u}|^2 d\Omega + \frac{\alpha}{2}\int_{\Gamma_c} |\mathbf{g}|^2 d\Gamma_c,$$

subject to

$$\begin{aligned}
(\mathbf{a}\cdot\nabla)\mathbf{u} - \nabla\cdot[-p\mathbf{I} + \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T)] &= 0 \quad \text{in} \quad \Omega,\\
\nabla\cdot\mathbf{u} &= 0 \quad \text{in} \quad \Omega,
\end{aligned}$$

$$\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma_c, \qquad \mathbf{u} = \mathbf{u}_{\text{in}} \quad \text{on } \Gamma_{\text{in}}, \qquad \mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega\backslash(\Gamma_c\cup\Gamma_{\text{in}}\cup\Gamma_{\text{out}}),$$

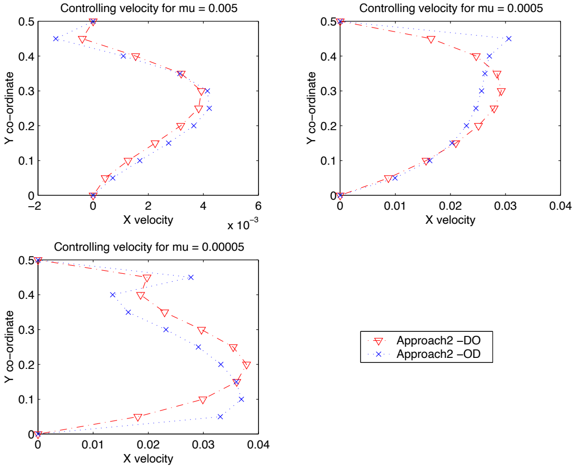
$$\mathbf{n}\cdot[-p\mathbf{I} + \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T)] = \mathbf{0} \quad \text{on } \Gamma_{\text{out}}.$$



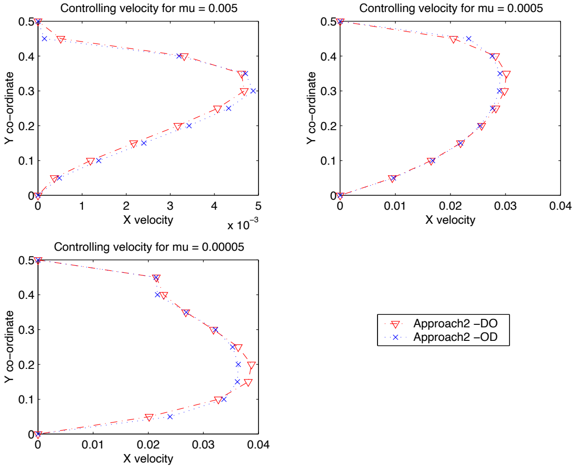
Note change of notation: $\mathbf{u}, p$ (velocities, pressure) states, $\mathbf{g}$ control.
Replace Dirichlet boundary condition $\mathbf{u} = \mathbf{g}$ on Γ_c by

$$\mathbf{n}\cdot[-p\mathbf{I} + \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T)] + 10^5\mathbf{u} = 10^5\mathbf{g} \text{ on } \Gamma_c.$$

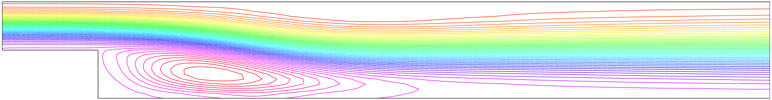
[Hou and Ravindran, 1998]



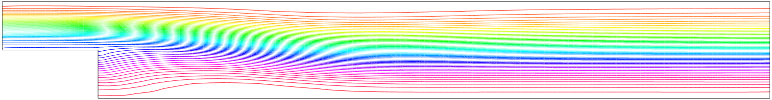
Optimal controls obtained using DO and OD, coarse discretization, $\alpha = 10^{-5}$.



Optimal controls obtained using DO and OD, fine discretization, $\alpha = 10^{-5}$.



Uncontrolled flow



Controlled flow

$$\mu = 5 * 10^{-4},\ \alpha = 10^{-5}$$

Elliptic Model Problem - Uncertainties in PDE

- Need to also apply discretization in random variables.
- Many more variations that for deterministic problem.
- Discretize-then-optimize (e.g., sample average approximation (SAA))
  vs.
  optimize-then-discretize (e.g., stochastic approximation).
- Will focus on sample average approximation (SAA).

Sample Average Approximation (SAA)

- For a given control $u \in \mathcal{U}$, the objective function involves expectation of

$$F = \mathcal{J} \circ S : \mathcal{U} \to L^1_{\mathbb{P}^\xi}(\Xi),$$
$$u \mapsto [F(u)](\cdot) = \frac{1}{2} q\big(y(u;\cdot), y(u;\cdot), \cdot\big) + c\big(y(u;\cdot), \cdot\big),$$

  where $y(u;\xi)$ solves state equation at sample ξ.

- Monte Carlo, quasi-Monte Carlo, multilevel/mult-fidelity Monte Carlo, and sparse-grid methods approximate

$$\begin{aligned} \mathbb{E}\big[F(u)\big] &= \int_\Xi [F(u)](\xi) \, \mathsf{d}\mathbb{P}^\xi(\xi) \\ &\approx \sum_{i=1}^{N} \zeta_i [F_{h_i}(u)](\xi_i) \\ &:= \sum_{i=1}^{N} \zeta_i \Big(\frac{1}{2} q\big(y_{h_i}(u;\xi_i), y_{h_i}(u;\xi_i), \xi_i\big) + c\big(y_{h_i}(u;\xi_i), \xi_i\big) \Big), \end{aligned}$$

  where $\zeta_i \in \mathbb{R}$ and $y_{h_i}(u;\xi_i)$ is an approximation of state at $\xi = \xi_i$.

Algebraic Form

- Algebraic representation of $y_{h_i}(u_h;\xi_i)$ at control $u_h \in \mathcal{U}_h$ given by

$$\mathbf{y}_i(\mathbf{u}) = \mathbf{A}_i^{-1}(\mathbf{b}_i - \mathbf{B}_i\mathbf{u}),$$

- Algebraic representation of SAA is

$$\sum_{i=1}^{N} \zeta_i \Big(\frac{1}{2} \mathbf{u}^\top \mathbf{B}_i^\top \mathbf{A}_i^{-\top} \mathbf{Q}_i \mathbf{A}_i^{-1} \mathbf{B}_i\ \mathbf{u} - \big(\mathbf{B}_i^\top \mathbf{A}_i^{-\top} \big(\mathbf{c}_i + \mathbf{Q}_i \mathbf{A}_i^{-1} \mathbf{b}_i \big) \big)^\top \mathbf{u} \Big)$$

$$+ \sum_{i=1}^{N} \zeta_i \Big(\frac{1}{2} \mathbf{b}_i^\top \mathbf{A}_i^{-\top} \mathbf{Q}_i \mathbf{A}_i^{-1} \mathbf{b}_i + \mathbf{c}_i^\top \mathbf{A}_i^{-1} \mathbf{b}_i \Big).$$

- Weights ζ_i
  - Monte Carlo and quasi-Monte Carlo set $\zeta_i = 1/N$
  - Multilevel/mult-fidelity Monte Carlo, and sparse-grid methods generate negative and positive weights!
    $\Rightarrow$ potential loss of convexity.

Optimality Conditions

- $\mathbf{A}_i \in \mathbb{R}^{n_{y,i} \times n_{y,i}}$, $i = 1, \ldots, N$, invertible, $\mathbf{R} \in \mathbb{R}^{n_u \times n_u}$ symmetric positive definite, $\mathbf{R} + \sum_{i=1}^N \zeta_i \mathbf{B}_i^\top \mathbf{A}_i^{-\top} \mathbf{Q}_i \mathbf{A}_i^{-1} \mathbf{B}_i$ be symmetric positive definite.
- SAA discretization has unique solution $\mathbf{u} \in \mathbb{R}^{n_u}$.
- $\mathbf{u} \in \mathbb{R}^{n_u}$ solves SAA discretization if and only if

$$\Big(\sum_{i=1}^N \zeta_i \mathbf{B}_i^\top \mathbf{A}_i^{-\top} \mathbf{Q}_i \mathbf{A}_i^{-1} \mathbf{B}_i + \mathbf{R}\Big)\, \mathbf{u} = \sum_{i=1}^N \zeta_i \mathbf{B}_i^\top \mathbf{A}_i^{-\top} \big(\mathbf{c}_i + \mathbf{Q}_i \mathbf{A}_i^{-1} \mathbf{b}_i\big),$$

and if and only if there exist

$$\vec{\mathbf{y}} = \begin{pmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_N \end{pmatrix}, \quad \vec{\boldsymbol{\lambda}} = \begin{pmatrix} \boldsymbol{\lambda}_1 \\ \vdots \\ \boldsymbol{\lambda}_N \end{pmatrix},$$

such that

$$\begin{pmatrix} \vec{\mathbf{Q}} & \mathbf{0} & \vec{\mathbf{A}}^\top \\ \mathbf{0} & \mathbf{R} & \vec{\mathbf{B}}^\top \\ \vec{\mathbf{A}} & \vec{\mathbf{B}} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \vec{\mathbf{y}} \\ \mathbf{u} \\ \vec{\boldsymbol{\lambda}} \end{pmatrix} = \begin{pmatrix} -\vec{\mathbf{c}} \\ \mathbf{0} \\ \vec{\mathbf{b}} \end{pmatrix},$$

where

- KKT conditions

$$\begin{pmatrix} \vec{\mathbf{Q}} & \mathbf{0} & \vec{\mathbf{A}}^\top \\ \mathbf{0} & \mathbf{R} & \vec{\mathbf{B}}^\top \\ \vec{\mathbf{A}} & \vec{\mathbf{B}} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \vec{\mathbf{y}} \\ \mathbf{u} \\ \vec{\boldsymbol{\lambda}} \end{pmatrix} = \begin{pmatrix} -\vec{\mathbf{c}} \\ \mathbf{0} \\ \vec{\mathbf{b}} \end{pmatrix},$$

where

$$\vec{\mathbf{A}} = \begin{pmatrix} \zeta_1 \mathbf{A}_1 & & \\ & \ddots & \\ & & \zeta_N \mathbf{A}_N \end{pmatrix}, \ \vec{\mathbf{Q}} = \begin{pmatrix} \zeta_1 \mathbf{Q}_1 & & \\ & \ddots & \\ & & \zeta_N \mathbf{Q}_N \end{pmatrix}, \ \vec{\mathbf{B}} = \begin{pmatrix} \zeta_1 \mathbf{B}_1 \\ \vdots \\ \zeta_N \mathbf{B}_N \end{pmatrix},$$

and

$$\vec{\mathbf{b}} = \big(\zeta_1 \mathbf{b}_1^\top, \ldots, \zeta_N \mathbf{b}_N^\top\big)^\top, \quad \vec{\mathbf{c}} = \big(\zeta_1 \mathbf{c}_1^\top, \ldots, \zeta_N \mathbf{c}_N^\top\big)^\top.$$

- Note coupling

$$\mathbf{R}\mathbf{u} + \vec{\mathbf{B}}^\top \vec{\boldsymbol{\lambda}} = \mathbf{R}\mathbf{u} + \sum_{i=1}^N \zeta_i \mathbf{B}_i \boldsymbol{\lambda}_i = \mathbf{0}.$$

- Requires modification of standard KKT iterative solvers. See, e.g., [Kouri and Ridzal, 2018, Section 5.2] and [Nobile and Vanzan, 2023], [Ciaramella et al., 2024].

Discretization Error Analyses

- Monte Carlo, quasi-Monte Carlo, multilevel/mult-fidelity Monte Carlo generate random samples
  $\Rightarrow$ Need to measure error in expectation or probability.
- Multilevel/mult-fidelity Monte Carlo and sparse-grid methods generate negative and positive weights!
  $\Rightarrow$ Need to assume convexity of SAA problems.
  Numerical observation indicates that this is often true for sparse-grid methods (once sparse-grid is sufficiently fine) [Kouri, 2012], [Kouri, 2014] , but no proof (yet).
- Recent error estimates for Monte Carlo and Galerkin FEM of elliptic linear-quadratic optimal control problems by [Milz, 2023a, Milz, 2023b, Milz, 2023c], [Milz and Ulbrich, 2024], and [Römisch and Surowiec, 2024].

Discretization Error Analyses (cont.)

- Error estimate for Monte Carlo and Galerkin FEM (piecewise constant controls, piecewise linear states) discretization of elliptic linear-quadratic optimal control problems [Milz, 2023b]:

  $$\mathbb{E}\left[\|u_{h,N} - u\|_{L^2(D)}^2\right] \leq c_1 h^2 + c_2/N$$

  For each $\delta \in (0, 1)$, with a probability $1 - \delta$,

  $$\|u_{h,N} - u\|_{L^2(D)} \leq \tilde{c}_1 h + \tilde{c}_2 \sqrt{2\ln(2/\delta)/N},$$

- Note sample size N typically limiting, not FEM mesh size h, i.e., $h < 1/\sqrt{N}$.

Literature I

[Abraham et al., 2004] Abraham, F., Behr, M., and Heinkenschloss, M. (2004).
The effect of stabilization in finite element methods for the optimal boundary control of the Oseen equations.
Finite Elem. Anal. Des., 41:229–251, DOI: 10.1016/j.finel.2004.06.001, http://dx.doi.org/10.1016/j.finel.2004.06.001.

[Babuška et al., 2007] Babuška, I., Nobile, F., and Tempone, R. (2007).
A stochastic collocation method for elliptic partial differential equations with random input data.
SIAM J. Numer. Anal., 45(3):1005–1034 (electronic).

[Benzi et al., 2005] Benzi, M., Golub, G. H., and Liesen, J. (2005).
Numerical solution of saddle point problems.
Acta Numerica, 14:1–137, DOI: 10.1017/S0962492904000212, https://doi.org/10.1017/S0962492904000212.

[Borzì and Schulz, 2009] Borzì, A. and Schulz, V. (2009).
Multigrid methods for PDE optimization.
SIAM Review, 51(2):361–395, DOI: 10.1137/060671590, https://doi.org/10.1137/060671590.

Literature II

[Borzì and Schulz, 2012] Borzì, A. and Schulz, V. (2012).
Computational Optimization of Systems Governed by Partial Differential Equations.
Computational Science and Engineering, Vol. 8. SIAM, Philadelphia, DOI: 10.1137/1.9781611972054, https://doi.org/10.1137/1.9781611972054.

[Ciaramella et al., 2024] Ciaramella, G., Nobile, F., and Vanzan, T. (2024).
A Multigrid Solver for PDE-Constrained Optimization with Uncertain Inputs.
J. Sci. Comput., 101(1):Paper No. 13, DOI: 10.1007/s10915-024-02646-7, https://doi.org/10.1007/s10915-024-02646-7.

[Collis and Heinkenschloss, 2002] Collis, S. S. and Heinkenschloss, M. (2002).
Analysis of the streamline upwind/Petrov Galerkin method applied to the solution of optimal control problems.
Technical Report TR02–01, Department of Computational and Applied Mathematics, Rice University, Houston, TX 77005–1892, DOI: 10.48550/arXiv.2411.0982, https://doi.org/10.48550/arXiv.2411.0982.
Also available as arXiv:2411.09828.

[Heinkenschloss and Kouri, 2025] Heinkenschloss, M. and Kouri, D. P. (2025).
Optimization problems governed by systems of pdes with uncertainties.
Acta Numerica, 3x:x–x, DOI: 10.1017/S0962492925000029, https://doi.org/10.1017/S0962492925000029.

Literature III

[Heinkenschloss and Leykekhman, 2010] Heinkenschloss, M. and Leykekhman, D. (2010).

Local error estimates for SUPG solutions of advection-dominated elliptic linear-quadratic optimal control problems.

SIAM J. Numer. Anal., 47(6):4607–4638, DOI: 10.1137/090759902, http://dx.doi.org/10.1137/090759902.

[Hinze et al., 2009] Hinze, M., Pinnau, R., Ulbrich, M., and Ulbrich, S. (2009).

Optimization with PDE Constraints, volume 23 of *Mathematical Modelling, Theory and Applications*.

Springer Verlag, Heidelberg, New York, Berlin, DOI: 10.1007/978-1-4020-8839-1, http://doi.org/10.1007/978-1-4020-8839-1.

[Hou and Ravindran, 1998] Hou, L. S. and Ravindran, S. S. (1998).

A penalized Neumann control approach for solving an optimal Dirichlet control problem for the Navier-Stokes equations.

SIAM J. Control Optim., 36(5):1795–1814, DOI: 10.1137/S0363012996304870, https://doi.org/10.1137/S0363012996304870.

Literature IV

[Kouri, 2012] Kouri, D. P. (2012).

An Approach for the Adaptive Solution of Optimization Problems Governed by Partial Differential Equations with Uncertain Coefficients.

PhD thesis, Department of Computational and Applied Mathematics, Rice University, Houston, TX, https://hdl.handle.net/1911/64617.

[Kouri, 2014] Kouri, D. P. (2014).

A multilevel stochastic collocation algorithm for optimization of PDEs with uncertain coefficients.

SIAM/ASA J. Uncertain. Quantif., 2(1):55–81, DOI: 10.1137/130915960, http://doi.org/10.1137/130915960.

[Kouri and Ridzal, 2018] Kouri, D. P. and Ridzal, D. (2018).

Inexact trust-region methods for PDE-constrained optimization.

In Antil, H., Kouri, D. P., Lacasse, M.-D., and Ridzal, D., editors, *Frontiers in PDE-constrained optimization*, volume 163 of *IMA Vol. Math. Appl.*, pages 83–121. Springer, New York, DOI: 10.1007/978-1-4939-8636-1_3, http://doi.org/10.1007/978-1-4939-8636-1_3.

Literature V

[Leykekhman, 2012] Leykekhman, D. (2012).

Investigation of commutative properties of discontinuous Galerkin methods in PDE constrained optimal control problems.

J. Sci. Comput., 53(3):483–511, ISSN: 0885-7474, DOI: 10.1007/s10915-012-9582-y, http://doi.org/10.1007/s10915-012-9582-y.

[Leykekhman and Heinkenschloss, 2012] Leykekhman, D. and Heinkenschloss, M. (2012).

Local error analysis of discontinuous Galerkin methods for advection-dominated elliptic linear-quadratic optimal control problems.

SIAM J. Numer. Anal., 50(4):2012–2038, DOI: 10.1137/110826953, http://doi.org/10.1137/110826953.

[Lions, 1971] Lions, J.-L. (1971).

Optimal Control of Systems Governed by Partial Differential Equations.

Die Grundlehren der mathematischen Wissenschaften, Band 170. Springer-Verlag, New York-Berlin.

Translated from the French by S. K. Mitter.

Literature VI

[Manzoni et al., 2021] Manzoni, A., Quarteroni, A., and Salsa, S. (2021).
Optimal Control of Partial Differential Equations—Analysis, Approximation, and Applications, volume 207 of *Applied Mathematical Sciences*.
Springer, Cham, DOI: 10.1007/978-3-030-77226-0,
https://doi.org/10.1007/978-3-030-77226-0.

[Milz, 2023a] Milz, J. (2023a).
Consistency of Monte Carlo estimators for risk-neutral PDE-constrained optimization.
Appl. Math. Optim., 87(3):Paper No. 57, 25, DOI: 10.1007/s00245-023-09967-3,
https://doi.org/10.1007/s00245-023-09967-3.

[Milz, 2023b] Milz, J. (2023b).
Reliable error estimates for optimal control of linear elliptic PDEs with random inputs.
SIAM/ASA J. Uncertain. Quantif., 11(4):1139–1163, DOI: 10.1137/22M1503889,
https://doi.org/10.1137/22M1503889.

Literature VII

[Milz, 2023c] Milz, J. (2023c).
Sample average approximations of strongly convex stochastic programs in Hilbert spaces.
Optim. Lett., 17(2):471–492, DOI: 10.1007/s11590-022-01888-4, https://doi.org/10.1007/s11590-022-01888-4.

[Milz and Ulbrich, 2024] Milz, J. and Ulbrich, M. (2024).
Sample size estimates for risk-neutral semilinear PDE-constrained optimization.
SIAM J. Optim., 34(1):844–869, DOI: 10.1137/22M1512636, https://doi.org/10.1137/22M1512636.

[Nobile and Vanzan, 2023] Nobile, F. and Vanzan, T. (2023).
Preconditioners for robust optimal control problems under uncertainty.
Numer. Linear Algebra Appl., 30(2):Paper No. e2472, 31, DOI: 10.1002/nla.2472, https://doi.org/10.1002/nla.2472.

[Pearson and Pestana, 2020] Pearson, J. W. and Pestana, J. (2020).
Preconditioners for Krylov subspace methods: an overview.
GAMM-Mitt., 43(4):e202000015, 35, DOI: 10.1002/gamm.202000015, https://doi.org/10.1002/gamm.202000015.

Literature VIII

[Quarteroni, 2009] Quarteroni, A. (2009).

Numerical Models for Differential Problems, volume 2 of *MS&A. Modeling, Simulation and Applications*.

Springer-Verlag Italia, Milan, DOI: 10.1007/978-88-470-1071-0, http://dx.doi.org/10.1007/978-88-470-1071-0.

Translated from the 4th (2008) Italian edition by Silvia Quarteroni.

[Römisch and Surowiec, 2024] Römisch, W. and Surowiec, T. M. (2024).

Asymptotic properties of Monte Carlo methods in elliptic PDE-constrained optimization under uncertainty.

Numer. Math., 156(5):1887–1914, DOI: 10.1007/s00211-024-01436-5, https://doi.org/10.1007/s00211-024-01436-5.

[Tröltzsch, 2010] Tröltzsch, F. (2010).

Optimal Control of Partial Differential Equations: Theory, Methods and Applications, volume 112 of *Graduate Studies in Mathematics*.

American Mathematical Society, Providence, RI, DOI: 10.1090/gsm/112, http://dx.doi.org/10.1090/gsm/112.

Literature IX

[Wathen, 2015] Wathen, A. J. (2015).
Preconditioning.
Acta Numerica, 24:329–376, DOI: 10.1017/S0962492915000021,
http://dx.doi.org/10.1017/S0962492915000021.