Games and Equilibria under Uncertainty: Existence and Computation

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Mathematical Preliminaries I

Proposition 1

Suppose $X \subseteq \mathbb{R}^n$ is a closed and convex set and f is a convex and C^1 on an open set containing X. Consider the convex optimization problem given by

$$\text{Opt} \qquad \underset{x}{\text{minimize}} \quad f(x) \qquad \text{subject to} \quad x \in X.$$

Then the following holds.

$$[\, x^* \text{ solves (Opt)} \,] \iff [\, x^* \text{ solves } VI(X, \nabla_x f) \,] \tag{1}$$

This necessitates defining the variational inequality problem VI(X, F).

Mathematical Preliminaries II

Definition 2 (Variational Inequality Problem VI(X, F))

Given a set $X \subseteq \mathbb{R}^n$ and a continuous map $F : X \to \mathbb{R}^n$, the variational inequality problem VI(X, F) requires finding an $x \in X$ such that

$$(y - x)^\top F(x) \geq 0, \qquad \forall y \in X. \tag{2}$$

We denote the solutions of VI(X, F) by SOL(X, F).

- The variational inequality problem can be cast as a generalized equation.
- A generalized equation is an extension of a standard nonlinear equation, given by $H(x) = 0$ where $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued map.

$$0 \in H(x).$$

- To restate VI(X, F) as a generalized equation, we define the normal cone of a set X at a point $\tilde{x}$. Recall that a set $C \subseteq \mathbb{R}^n$ is called a cone if $\lambda x \in C$ for any $x \in C$ and $\lambda \geq 0$.

Mathematical Preliminaries III

Definition 3 (Normal cone $\mathcal{N}_X(x)$)

Given a set $X \subseteq \mathbb{R}^n$, the normal cone of a set X at $\tilde{x}$ is defined as

$$\mathcal{N}_X(\tilde{x}) \triangleq \left\{ d \in \mathbb{R}^n \mid d^\top (y - \tilde{x}) \leq 0, \, \forall \, y \in X \right\}. \tag{3}$$

Any element of $\mathcal{N}_X(x)$ is called a normal vector to X at x.

When X is a convex set,

$$[\, x \text{ solves VI}(X, F) \,] \iff [\, -F(x) \in \mathcal{N}_X(x) \,] \equiv \left[0 \in H(x) \triangleq F(x) + \mathcal{N}_X(x) \right].$$

The problem VI(X, F) can be seen to capture a range of equilibrium problems, including, of course, optimization problems.

Mathematical Preliminaries IV

Lemma 4 (Root-finding and saddle-point problems)

(a) (Root-finding problem). Suppose $X = \mathbb{R}^n$. *Then* $x \in SOL(X, F)$ *if and only if* x *is a zero of the mapping* F, *i.e.* $F(x) = 0$.

(b) (Saddle-point problem). Suppose X *and* Y *are closed and convex sets and* L *is a smooth convex-concave function on* $Z \triangleq X \times Y$. *Then the following holds.*

$$(x, y) \text{ solves } \left\{ \min_{x \in X} \max_{y \in Y} L(x, y) \right\} \iff (x, y) \text{ solves } VI(Z, F)$$

$$\text{where } Z \triangleq X \times Y \text{ and } F(x, y) \triangleq \begin{pmatrix} \nabla_x L(x, y) \\ -\nabla_y L(x, y) \end{pmatrix}.$$

When X is a cone, VI$(X, F) \equiv$ a *complementarity problem* CP(X, F).

Definition 5 (Complementarity Problem CP(X, F))

Mathematical Preliminaries V

Given a cone X and a mapping $F : X \to \mathbb{R}^n$, the complementarity problem CP(X, F) requires finding an $x \in \mathbb{R}^n$ such that

$$X \ni x \perp F(x) \in X^*,$$

where $u \perp v$ implies that $u^\top v = 0$ and X^* is the dual cone of X, defined as

$$X^* \triangleq \left\{ v \mid v^\top d \geq 0, \quad \forall d \in X \right\}. \tag{4}$$

The orthogonality requirement $x^\top F(x)$ can be expressed as componentwise products as being zero, i.e. $x_i F_i(x) = 0, \forall, i \in [n]$.

In fact, VI(X, F) $\equiv$ CP(X, F), when X is a cone.

Proposition 6 (Equivalence between VI(X, F) and CP(X, F))

Mathematical Preliminaries VI

Let X be a cone in $\mathbb{R}^n$.

$$[\, x \textit{ solves } VI(X, F)\,] \iff [\, x \textit{ solves } CP(X, F)\,]. \tag{5}$$

There are some important special cases of CP(X, F) that deserve further discussion as they appear later in our course.

Mathematical Preliminaries VII

Definition 7 (Examples of CPs)

Given a mapping $F : X \to \mathbb{R}^n$ and a cone $X \subseteq \mathbb{R}^n$.

(a) If $X \triangleq \mathbb{R}^n_+$, then CP($X, F$) reduces to the nonlinear complemenarity problem NCP(F), defined as

$$0 \leq x \perp F(x) \geq 0.$$

(b) If $X \triangleq \mathbb{R}^n_+$ and $F(x) = Mx + q$, where $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, CP(X, F) reduces to the linear complemenarity problem LCP(M, q), defined as

$$0 \leq x \perp Mx + q \geq 0.$$

Mathematical Preliminaries VIII

Lemma 8

(a) (Linear program). Then the following holds.

$$(x,\lambda) \text{ solves } \left\{ \begin{array}{c} \min\ c^\top x \\ \text{subject to } Ax \geq b \quad (\lambda) \\ x \geq 0 \end{array} \right\} \iff (x,\lambda) \text{ solves } LCP(q,M)$$

$$\text{where } M \triangleq \begin{pmatrix} \mathbf{0} & -A^\top \\ A & 0 \end{pmatrix} \text{ and } q \triangleq \begin{pmatrix} c \\ -b \end{pmatrix}.$$

(b) (Quadratic program). Then the following holds.

$$(x,\lambda) \text{ solves } \left\{ \begin{array}{c} \min\ \frac{1}{2} x^\top Q x + c^\top x \\ \text{subject to } Ax \geq b \quad (\lambda) \\ x \geq 0 \end{array} \right\} \iff (x,\lambda) \text{ solves } LCP(q,M)$$

$$\text{where } M \triangleq \begin{pmatrix} Q & -A^\top \\ A & 0 \end{pmatrix} \text{ and } q \triangleq \begin{pmatrix} c \\ -b \end{pmatrix}.$$

Continuous-strategy NEPs I

Definition 9 (Noncooperative games $\mathscr{G}$)

The class of games $\mathscr{G}$ consists of N-player noncooperative games comprising of a collection of players, indexed by i. Suppose player i is characterized by strategy set $\mathcal{X}_i$ and objective $f_i(\bullet, \mathbf{x}^{-i})$, defined as $f_i(x^i, \mathbf{x}^{-i})$, where $x^i \in \mathcal{X}_i \subseteq \mathbb{R}^{n_i}$, $n \triangleq \sum_{i=1}^{N} n_i$, and $\mathbf{x}^{-i} \triangleq (x^j)_{j \neq i}$.

A Nash equilibrium (NE) of a noncooperative game $\mathcal{G} \in \mathscr{G}$ is a tuple $\mathbf{x}^* \triangleq \{x^{1,*}, \cdots, x^{N,*}\}$, where

$$x^{i,*} \in \operatorname*{argmin}_{x^i \in \mathcal{X}_i} f_i(x^i, \mathbf{x}^{-i,*}), \quad \forall i \in [N]. \tag{NE}$$

- An NE is a set of decisions at which no player has a desire to unilaterally deviate [Nash, 1950].

Continuous-strategy NEPs II

- In some instances, we require articulating an ϵ-Nash equilibrium, denoted by $\mathbf{x}_\epsilon^* \triangleq \{x_\epsilon^{1,*}, \cdots, x_\epsilon^{N,*}\}$, such that ϵ-optimality holds for for the ith player's problem for $i \in [N]$, i.e.

$$f_i(x_\epsilon^{i,*}, \mathbf{x}_\epsilon^{-i,*}) \leq \min_{x^i \in \mathcal{X}_i} f_i(x^i, \mathbf{x}_\epsilon^{-i,*}) + \epsilon, \quad \forall\, i \in [N]. \qquad (\epsilon\text{-NE})$$

Smooth Convex Games I

Definition 10 (**Smooth convex games** $\mathscr{G}^{sc}$)

The class of smooth convex games $\mathscr{G}^{\mathrm{sc}} \subseteq \mathscr{G}$ contains N-player games in which for any $i \in \mathcal{N}$, $\mathcal{X}_i$ is a closed and convex set in $\mathbb{R}^{n_i}$ and for any $\mathbf{x}^{-i} \in \prod_{j=1, j\neq i}^{n} \mathcal{X}_j$, $f_i(\bullet, \mathbf{x}^{-i})$ is convex and continuously differentiable on an open set containing $\mathcal{X}_i$.

- Specifically, if for any $\mathbf{x}^{-i,*} \in \prod_{j\neq i} \mathcal{X}_j$, $f_i(x^i, \mathbf{x}^{-i,*})$ is C^1 and convex in x^i on an open set containing $\mathcal{X}_i$ and $\mathcal{X}_i$ is a closed and convex set, then any minimizer $x^{i,*}$ satisfies the following variational inequality problem.

$$(\tilde{x}^i - x^{i,*})^\top \nabla_{x^i} f_i(x^{i,*}, \mathbf{x}^{-i,*}) \geq 0, \qquad \forall \tilde{x}^i \in \mathcal{X}_i. \tag{6}$$

- Consequently, the necessary and sufficient conditions for an equilibrium $\{x^{1,*}, \cdots, x^{N,*}\}$ are (6) for $i = 1, \cdots, N$.

Smooth Convex Games II

- In fact, it can be shown [Facchinei and Pang, 2003] that $\mathbf{x}^* \triangleq \{x^{1,*}, \cdots, x^{N,*}\}$ satisfies (6) for $i = 1, \cdots, N$ if and only if $\mathbf{x}^*$ is a solution of VI($\mathcal{X}, F$), where $\mathcal{X}$ and F are defined as follows.

$$F(\mathbf{x}) \triangleq \begin{pmatrix} \nabla_{x^1} f_1(x^1, \mathbf{x}^{-1}) \\ \vdots \\ \nabla_{x^N} f_N(x^N, \mathbf{x}^{-N}) \end{pmatrix} \text{ and } \mathcal{X} \triangleq \prod_{i=1}^{N} \mathcal{X}_i, \tag{7}$$

- Equivalence between computing an (NE) of a game lying in ($\mathscr{G}^{sc}$) and solving a suitably defined variational inequality problem.

Theorem 11 (NE of $\mathscr{G}^{\rm sc}$ and Solutions of VI($\mathcal{X}, F$))

Consider an N-player noncooperative game in $\mathscr{G}^{\rm sc}$. Suppose $\mathcal{X}$ and F are as defined in (7)*. Then the following holds.*

$$[\, \mathbf{x} \textit{ is an NE }] \iff [\, \mathbf{x} \in SOL(X, F)\,].$$

Smooth Convex Games III

Example 12 (Nash-Cournot equilibrium problem (NCEP))

Consider a setting where a collection of N players sell a product. Suppose the ith player is characterized by a convex and twice continuously differentiable production cost $c_i(x^i)$ and capacities Cap_i while the inverse demand function p is a non-increasing, concave, and twice continuously differentiable function in X where $X \triangleq \sum_{i=1}^{N} x^i$. If $\mathcal{X}_i \triangleq \{x^i \mid 0 \leq x^i \leq \text{Cap}_i\}$ for $i \in [N]$, then a Nash-Cournot equilibrium is a tuple $\{x^{1,*}, \cdots, x^{N,*}\}$ such that

$$x^{i,*} \in \operatorname*{argmin}_{x^i \in \mathcal{X}_i} \left(c_i(x^i) - p(x^i + X^{-i})x^i\right), \qquad i = 1, \cdots, N. \tag{8}$$

We may observe that the associated noncooperative game lies in $\mathscr{G}^{\text{sc}}$, since $c_i(x^i) - p(X)x_i$ is a smooth convex function in x_i for any $\mathbf{x}_{-i} \in \prod_{j \neq i} \mathcal{X}_j$. By Theorem 11, $\mathbf{x}^*$ is a Nash-Cournot equilibrium if and only if $\mathbf{x}^*$ is a solution to $VI(\mathcal{X}, F)$, where

$$F(\mathbf{x}) \triangleq \begin{pmatrix} c_1'(x_1) - p'(X)x_1 - p(X) \\ \vdots \\ c_N'(x_1) - p'(X)x_N - p(X) \end{pmatrix}.$$

Nonsmooth convex games I

We now consider the subclass of nonsmooth convex games $\mathscr{G}^{\rm nc}$, where player problems are convex but player objectives are not necessarily smooth.

Definition 13 (Nonsmooth convex games $\mathscr{G}^{\rm nc}$)

The class of nonsmooth convex games $\mathscr{G}^{\rm nc} \subseteq \mathscr{G}$ contains N-player games in which for any $i \in \mathcal{N}$, $\mathcal{X}_i$ is a closed and convex set in $\mathbb{R}^{n_i}$ and any $\mathbf{x}^{-i}$, player i's objective $f_i(\cdot, \mathbf{x}^{-i})$ is assumed to be a continuous and convex function on an open set containing $\mathcal{X}_i$.

- Since $f_i(\bullet, \mathbf{x}^{-i})$ admits a subdifferential at x^i denoted by $\partial_{x^i} f_i(x^i, \mathbf{x}^{-i})$, given any $\mathbf{x}^{-i,*}$ and any $v_i \in \partial_{x^i} f_i(x^i, \mathbf{x}^{-i,*})$,

$$(\tilde{x}^i - x^{i,*})^\top v_i \geq 0, \quad \forall \tilde{x}_i \in \mathcal{X}_i. \tag{9}$$

Nonsmooth convex games II

- $\left[\{x^{1,*}, \cdots, x^{N,*}\} \text{ solves (9) for } i \in [N]\right] \iff [\mathbf{x}^* \text{ solves VI}(\mathcal{X}, \Phi)]$, where Φ is a set-valued map defined as follows.

$$\Phi(\mathbf{x}) \triangleq \prod_{i=1}^{N} \partial_{x^i} f_i(x^i, \mathbf{x}^{-i}). \tag{10}$$

- We define the *generalized* variational inequality problem in which the map $\Phi : \mathcal{X} \rightrightarrows \mathbb{R}^n$ is a set-valued map.

Definition 14 (Generalized variational inequality problem VI(Φ, X))

Given a set $X \subseteq \mathbb{R}^n$ and a map $\Phi : X \rightrightarrows \mathbb{R}^n$ is a set-valued map. Then x is a solution of VI(X, Φ) if there exists an x and a vector $u \in \Phi(x)$ such that

$$(y - x)^\top u \geq 0, \qquad \forall y \in X. \tag{11}$$

Nonsmooth convex games III

It can be seen that $x^{i,*}$ is a minimizer of $f_i(\bullet, x^{-i,*})$ over $\mathcal{X}_i$ if and only if $x^{i,*}$ is a solution of VI$(\mathcal{X}_i, \partial_{x^i} f_i(\bullet, x^{-i,*}))$. This allows us to derive a relationship between the (NE) of a game in $\mathscr{G}^{nc}$ and a solution of a variational inequality problem with a set-valued map is presented next.

Theorem 15 (NE of $\mathscr{G}^{nc}$ and Solutions of VI$(\mathcal{X}, \Phi)$)

Consider an N-player noncooperative game in $\mathscr{G}^{\rm nc}$. Suppose $\mathcal{X}$ and Φ are defined as (7) *and* (10)*, respectively. Then the following holds.*

$$[\,\mathbf{x} \text{ is an NE }] \iff [\,\mathbf{x} \in SOL(X, \Phi)\,].$$

We now apply this result to a nonsmooth variant of NCEP.

Nonsmooth convex games IV

Example 16 (Nonsmooth Nash-Cournot equilibrium problem (NCEP))

Consider Example 12 where for any $i \in \mathcal{N}$, $c_i(x^i)$ is assumed to be convex and continuous over an open set containing X_i. We may observe that the associated noncooperative game lies in $\mathscr{G}^{\rm nc}$, since $c_i(x^i) - p(X)x_i$ is a nonsmooth convex function in x^i for any $x^{-i} \in \prod_{j=1,j\neq i}^{N} \mathcal{X}_j$. By Theorem 15, $\mathbf{x}^*$ is a Nash-Cournot equilibrium if and only if $\mathbf{x}^*$ is a solution to VI($\mathcal{X}$, Φ), where

$$\Phi(\mathbf{x}) \triangleq \left[\prod_{i=1}^{N} \partial_{x^i} c_i(x^i)\right] - p'(X)x - p(X)\mathbf{1}.$$

Monotone games I

Definition 17 (Monotone maps)

Consider a single-valued continuous map $F : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$. F is said to be
(a) (strictly) monotone on X if

$$(F(x) - F(y))^\top (x - y)\,(>) \geq 0, \qquad \forall\, x, y, \in X. \tag{12}$$

(b) strongly monotone on X if there exists a $c > 0$,

$$(F(x) - F(y))^\top (x - y) \geq c\|x - y\|^2, \qquad \forall\, x, y, \in X. \tag{13}$$

A set-valued map $\Phi : X \rightrightarrows \mathbb{R}^n$ is monotone if

$$(x - y)^\top (u - v) \geq 0, \tag{14}$$

for all $x, y \in X$ and for all $u \in \Phi(x)$ and $v \in \Phi(y)$.

Monotone games II

- Consider the following optimization problem in which $f : X \to \mathbb{R}^n$ is convex and C^1 on an open set containing X, a closed and convex set.

$$\begin{aligned} \min_{x} \quad & f(x) \\ \text{subject to} \quad & x \in X. \end{aligned} \tag{15}$$

- Then $\nabla_x f$ is a monotone map on X, i.e.

$$(\nabla_x f(x) - \nabla_x f(y))^\top (x - y) \geq 0, \qquad \forall\, x, y, \in X. \tag{16}$$

- Similarly, if f is strictly convex on X, then $\nabla_x f$ is strictly monotone, i.e.

$$(\nabla_x f(x) - \nabla_x f(y))^\top (x - y) > 0, \qquad \forall\, x, y, \in X. \tag{17}$$

- Next, we consider the class of monotone games where the map of the equivalent variational inequality problem (either F or Φ) is monotone.

Monotone games III

Definition 18 (**Monotone games)**

The class of monotone games $\mathscr{G}^{\rm mn} \subseteq (\mathscr{G}^{\rm sc} \cup \mathscr{G}^{\rm nc}) \subseteq \mathscr{G}$ contains N-player games in which the associated map in the variational inequality problem (either F or Φ) is monotone on $\mathcal{X}$.

Consider Example 16. We proceed to show that the mapping Φ is monotone on $\mathcal{X}$ if $p(X) \triangleq a - bX$. Let $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{X}$ and let $\phi_1 \in \Phi(\mathbf{y}_1)$ and $\phi_2 \in \Phi(\mathbf{y}_2)$. By the definition of $\Phi(\mathbf{x})$, we have that if $\phi \in \Phi(\mathbf{x})$, it follows that

$$\phi = \phi^c + b(\mathbf{I} + \mathbf{1}\mathbf{1}^\top)\mathbf{x} - a\mathbf{1}, \text{ where } \phi^c = \left(\phi^{i,c}\right)_{i=1}^N \text{ and } \phi^{i,c} \in \partial_{x^i} c_i(x^i).$$

Then we have that

$$\begin{aligned}(\phi_1 - \phi_2)^\top(\mathbf{y}_1 - \mathbf{y}_2) &= \sum_{i=1}^N \underbrace{(\phi_1^{i,c} - \phi_2^{i,c})^\top(\mathbf{y}_1^i - \mathbf{y}_2^i)}_{(**) \geq 0} + b(\mathbf{y}^1 - \mathbf{y}^2)^\top(\mathbf{I} + \mathbf{1}\mathbf{1}^\top)(\mathbf{y}_1 - \mathbf{y}_2) \\ &\geq b\|\mathbf{y}_1 - \mathbf{y}_2\|^2,\end{aligned}$$

Monotone games IV

where the (**) inequality follows from the monotonicity of the subdifferential map and the first inequality is a consequence of $v^\top(I + ee^\top)v \geq b\|v\|^2$ for any v.

Potential games I

- We initiate our discussion by considering the following problem.

$$\min_{\mathbf{x} \in \mathcal{X}} \; h(\mathbf{x}), \tag{Opt}$$

where h is a C^1 on an open set containing $\mathcal{X}$, a closed and convex set.

- If $\mathbf{x}$ is a local minimizer of (Opt), then

$$(\tilde{\mathbf{x}} - \mathbf{x})^\top \nabla_x h(\mathbf{x}) \geq 0, \qquad \forall \tilde{\mathbf{x}} \in \mathcal{X}.$$

Specifically, if $\mathbf{x}$ is a local minimizer of (Opt), then $\mathbf{x}$ solves VI($\mathcal{X}, \nabla_x h$).

- In fact, if one imposes an additional assumption of convexity on h, then $\mathbf{x}$ is a solution of (Opt) if and only if $\mathbf{x}$ solves VI($\mathcal{X}, \nabla_x h$).
- Is computing a solution to VI($\mathcal{X}, F$) always reducible to obtaining a stationary point of a suitably defined optimization problem?
- This holds when there exists an h such that $F(\mathbf{x}) = \nabla_x h(\mathbf{x})$ over the domain of F. This is clarified by the next result.

Potential games II

Theorem 19

Consider VI($\mathcal{X}$, Φ) associated with a game in $\mathscr{G}$. Suppose the mapping Φ can be expressed as $\Phi(\mathbf{x}) \triangleq G(\mathbf{x}) + \prod_{i=1}^{N} \partial_{x^i} c_i(\mathbf{x}^i)$, where c_i is continuous and convex function defined on an open set containing $\mathcal{X}_i$ for $i = 1, \cdots, N$ and G is a continuously differentiable map on an open set containing $\mathcal{X}$ with a symmetric Jacobian $\nabla_x G(\mathbf{x})$ matrix for every $\mathbf{x} \in \mathcal{X}$. Then the following hold.

- (i) *There exists a real-valued function h such that $G(\mathbf{x}) = \nabla_x h(\mathbf{x})$ for all $\mathbf{x} \in dom(G)$.*
- (ii) *The function h is given by $h(\mathbf{x}) = \int_0^1 G(\mathbf{x}^0 + t(\mathbf{x} - \mathbf{x}^0))^\top (\mathbf{x} - \mathbf{x}^0) dt$ for any arbitrary $\mathbf{x}^0 \in \mathcal{X}$.*
- (iii) *The variational inequality problem VI($\mathcal{X}$, Φ) represents the stationarity conditions of (Opt$^{\mathscr{G}}$).*

$$\min_{\mathbf{x} \in \mathcal{X}} \left(h(\mathbf{x}) + \sum_{i=1}^{N} c_i(x_i) \right). \qquad (\text{Opt}^{\mathscr{G}})$$

This result allows for obtaining an equilibrium of the relevant game by obtaining a stationary point of (Opt$^{\mathscr{G}}$). We provide an example of this result by revisiting the nonsmooth Nash-Cournot equilibrium problem.

Potential games III

Example 20 (**Nash-Cournot equilibrium problem)**

Consider Example 16 and VI($\mathcal{X}$, Φ). By definition, $\Phi(\mathbf{x}) = G(\mathbf{x}) + \prod_{i=1}^{n} \partial_{x^i} c_i(x_i)$, where $G(\mathbf{x}) \triangleq b(I + ee^\top)\mathbf{x} - ae$.

Since $\mathbf{J}_x G(\mathbf{x}) = b(I + ee^\top)$, a constant and symmetric matrix, we may invoke Theorem 19. It follows that a Nash-Cournot equilibrium can be obtained through the solution of the following problem.

$$\min_{\mathbf{x} \in \mathcal{X}} \quad \left[\frac{b}{2} \mathbf{x}^\top (\mathbf{I} + \mathbf{1}\mathbf{1}^\top)\mathbf{x} - a\mathbf{1}^\top \mathbf{x} + \sum_{i=1}^{N} c_i(x^i) \right]. \tag{NCOPT}$$

Definition 21 (**Potential and weighted potential games $\mathscr{G}^{\rm pot}$)**

Consider the subclass of N-player weighted potential games $\mathscr{G}^{\rm pot} \subset \mathscr{G}$. Associated with any game $\mathcal{G} \in \mathscr{G}^{\rm pot}$ is a weighted potential function $h(\mathbf{x})$ such that the following holds for any $i \in \mathcal{N}$ and any $x^i, y^i \in \mathcal{X}_i$ and positive scalars $w_1, \cdots, w_N$.

$$w_i(h(x^i, \mathbf{x}^{-i}) - h(y^i, \mathbf{x}^{-i})) = f_i(x^i, \mathbf{x}^{-i}) - f_i(y^i, \mathbf{x}^{-i}). \tag{18}$$

If $w_i = 1$ for every i, then $\mathcal{G}$ is referred to as a potential game. □

Potential games IV

In fact, when the objective functions are continuously differentiable, condition (18) for $\mathbf{w} \equiv \mathbf{1}$ is equivalent to the requirement that for any $\mathbf{x}^{-i} \in \mathbf{X}_{-i}$, we have that

$$\nabla_{x^i} f_i(x^i, \mathbf{x}^{-i}) = \nabla_{x^i} h(x^i, \mathbf{x}^{-i}). \tag{19}$$

In Monderer and Shapley (1996), a necessary and sufficient condition for the potentiality of a game has been provided under the twice continuous differentiability of player objectives.

Potential games V

Proposition 22 (**Potentiality under C^2 objectives)**

Suppose for $i = 1, \cdots, N$, $f_i(\mathbf{x})$ is twice continuously differentiable in $\mathbf{x}$ on an open set containing $\mathcal{X}$, where $\mathcal{X}$ is a rectangle. Then $\mathcal{G} \in \mathscr{G}^{\rm sc}$ is a potential game if and only if

$$\nabla^2_{x_i,x_j} f_i(\mathbf{x}) = \nabla^2_{x_j,x_i} f_j(\mathbf{x}) \text{ for every } i \in \mathcal{N} \text{ and } \forall \mathbf{x} \in \mathcal{X}. \tag{20}$$

Furthermore, if the player objectives satisfy (20) and $\mathbf{x}_0 \in \mathcal{X}$, then a potential for $\mathcal{G}$ is given by

$$P(\mathbf{x}) \triangleq \sum_{i=1}^{N} \int_0^1 \frac{\partial f_i(\tilde{\mathbf{x}}(t))}{\partial x_i} \tilde{x}_i(t) dt,$$

where $\tilde{\mathbf{x}} : [0,1] \to \mathcal{X}$ is a piecewise continuously differentiable path in $\mathcal{X}$ that connects $\mathbf{x}^0$ to $\mathbf{x}^1$. □

Potential games VI

We conclude this section with our main result. A Nash equilibrium of a potential game is obtainable as a stationary point of the optimization problem

$$\min_{\mathbf{x} \in \mathcal{X}} h(\mathbf{x}). \qquad \text{(Pot-Opt)}$$

Theorem 23

Consider an N-player noncooperative game $\mathcal{G}$. Suppose for $i \in [N]$, $\mathcal{X}_i$ is closed, convex, and compact. For every $i \in [N]$, $f_i(\bullet, \mathbf{x}^{-i})$ is a convex and C^1 function on an open set $\mathcal{O}_i \supseteq \mathcal{X}_i$ for any $\mathbf{x}^{-i} \in \mathcal{X}^{-i}$. Finally, there exists a potential function h such that for any $i \in [N]$, (18) *for any $\mathbf{x}^{-i} \in \mathbf{X}^{-i}$. Then the following holds.*

$$[\, x \text{ is an NE }] \iff [\, x \text{ is a stationary point of (Pot-Opt)}]. \qquad (21)$$

Stochastic Optimization I

- Consider a decision-maker faced by a cost function ψ dependent on her decision $x \in X$ and a random variable $\boldsymbol{\xi}$, defined as $\boldsymbol{\xi} : \Omega \to \mathbb{R}^d$, where $(\Omega, \mathcal{F}, \mathbb{P})$ denotes the associated probability space.
- Suppose Ξ is given by $\{ \xi(\omega) \mid \omega \in \Omega \}$.
- For ease of exposition, we refer to $\psi(x, \xi(\omega))$ as $\psi(x, \xi)$ where ψ is a real-valued function given by $\psi : \mathcal{O} \times \Omega \to \bar{\mathbb{R}}$.
- A function $\psi : \mathcal{O} \times \Omega \to \bar{\mathbb{R}}$ is said to be a *random* function if for every $x \in X$, $\psi(x, \bullet)$ is $\mathcal{F}$-measurable.
- It then makes sense for the decision-maker to consider minimizing the expected value function θ, where

$$\theta(x) \triangleq \mathbb{E}[\psi(x, \boldsymbol{\xi})] = \int_\Omega \psi(x, \xi) d\mathbb{P}. \tag{22}$$

The resulting optimization problem may then be defined as follows.

$$\underset{x \in X}{\textbf{minimize}}\ \theta(x) \triangleq \mathbb{E}[\psi(x, \boldsymbol{\xi})]. \tag{23}$$

Stochastic Optimization II

- Recall that the function θ is said to be well defined on $X \subseteq \mathbb{R}^n$ if for every $x \in X$, either $\mathbb{E}\,[\,[\psi(x, \boldsymbol{\xi})]_+\,] < \infty$ or $\mathbb{E}\,[\,[-\psi(x, \boldsymbol{\xi})]_+\,] < \infty$.
- The expectation function θ inherits a host of properties from the function $\psi(\bullet, \xi)$ for $\omega \in \Omega$, as stated in Theorem 24.

Stochastic Optimization III

Theorem 24

Consider the random function $\psi : \mathbb{R}^n \times \Omega \to \bar{\mathbb{R}}$.
(a) Suppose for almost every $\omega \in \Omega$, $\psi(\bullet, \xi)$ *is a lower semicontinuous at a point* x_0 *and there exists an integrable function* $Z : \Omega \to \mathbb{R}$ *such that* $\psi(x, \xi) \geq Z(\xi)$ *for almost every* $\omega \in \Omega$ *and for all* x *in a neighborhood of* x_0.

(b) Suppose for almost every $\omega \in \Omega$, $\psi(\bullet, \xi)$ *is a convex function.*

(c) Suppose θ *is well defined and finite-valued at a given point* x_0. *There exists a positive-valued random variable* $Lip : \Omega \to \mathbb{R}_{++}$ *such that* $\mathbb{E}\,[\,Lip(\boldsymbol{\xi})\,] < +\infty$ *and for all* x_1, x_2 *in a neighborhood of* x_0 *and for almost every* $\omega \in \Omega$, $|\psi(x_1, \xi) - \psi(x_2, \xi)| \leq Lip(\xi)\|x_1 - x_2\|$.
(d) For almost every $\omega \in \Omega$, $\psi(\bullet, \xi)$ *is directionally differentiable at* x_0.

(e) For almost every $\omega \in \Omega$, $\psi(\bullet, \xi)$ *is differentiable at* x_0.

Then the following hold.

(i) If (a) holds, then for all x *in a neighborhood of* x_0, θ *is well defined and lower semicont. in a nbhd of* x_0.

(ii) If (b) holds, then θ *is a convex function;* *(iii) If (c) holds, then* θ *is locally Lipschitz at* x_0.

(iv) If (c) and (d) hold, then θ *is directionally differentiable at* x_0 *and* $\theta'(x_0, h) = \mathbb{E}\left[\,\psi'(x_0, \boldsymbol{\xi}, h)\,\right]$ *for all* $h \in \mathbb{R}^n$.

(v) If (c) and (e) hold, then θ *is differentiable at* x_0 *and* $\nabla_x \theta(x_0) = \mathbb{E}\,[\,\nabla_x \psi(x_0, \boldsymbol{\xi})\,]$.

Stochastic Optimization IV

- If X is a convex set and θ is differentiable at a local minimizer $\bar{x}$ of (23),

$$\nabla\theta(\bar{x})^\top(x-\bar{x}) \geq 0, \quad \forall\, x \in X. \tag{24}$$

- Under conditions (c) and (e) of Theorem 24, we may interchange derivatives and expectations, i.e.

$$\nabla_x\theta(\bar{x}) = \mathbb{E}\left[\,\nabla_x\psi(x,\boldsymbol{\xi})\,\right], \tag{25}$$

where the expectation of a vector, as denoted by $\mathbb{E}\left[\nabla_x\psi(x,\boldsymbol{\xi})\right]$, is implied in a component-wise fashion, i.e.

$$\mathbb{E}\left[\,\nabla_x\psi(x,\boldsymbol{\xi})\,\right] \triangleq \begin{pmatrix} \mathbb{E}\left[\nabla_{x_1}\psi(x,\boldsymbol{\xi})\right] \\ \vdots \\ \mathbb{E}\left[\,\nabla_{x_n}\psi(x,\boldsymbol{\xi})\,\right] \end{pmatrix}. \tag{26}$$

- Note that (24) is a special case of a stochastic variational inequality problem, a generalization of the deterministic variational inequality problem, where the components of the map are expectation-valued.

Stochastic Optimization V

Definition 25 (Stochastic VI and CP)

Let $F : \mathcal{O} \to \mathbb{R}^n$ and $\tilde{F} : \mathcal{O} \times \Omega \to \mathbb{R}^n$ be vector functions, where $\mathcal{O}$ is an open set containing the closed set X and $F_i(x) = \mathbb{E}\left[\tilde{F}_i(x, \omega)\right]$ for $i = 1, \ldots, n$.

The stochastic variational inequality problem (SVI), defined by the pair $(\tilde{F}, X)$, is to find a vector $\bar{x} \in X$ such that

$$\mathbb{E}\left[\tilde{F}(\bar{x}, \xi)\right]^\top (x - \bar{x}) \geq 0, \quad \forall\, x \in X.$$

SVI$(\mathcal{E}, F, X)$ is equivalent to the *stochastic complementarity problem* (SCP):

$$X \ni \bar{x} \perp \mathbb{E}\left[\tilde{F}(\bar{x}, \xi)\right] \in X^*, \tag{27}$$

when X is additionally a cone and X^* is its dual. □

Stochastic Optimization VI

- In a *single-stage* stochastic optimization model, a decision x is made prior to the revelation of uncertainty and the decision-maker selects an x that minimizes the expected outcome $\mathbb{E}[\,\psi(x, \boldsymbol{\xi})\,]$ over a feasible set X.

Stochastic Optimization VII

Example 26 (**Single-stage newsvendor problem)**

Consider a newsvendor faced by ordering decision x and faced by uncertain demand, embodied by random variable $\boldsymbol{\xi}$, where $\boldsymbol{\xi} : \Omega \to \mathbb{R}_{++}$ and $(\Omega, \mathcal{F}, \mathbb{P})$ represents the associated probability space. Suppose the unit cost of newspapers is denoted by $c > 0$ while the unit back-order and storage costs are $b > 0$ and $h > 0$, respectively. Consequently, the random cost function $\psi(\bullet, \xi(\omega))$ is defined as

$$\psi(x, \xi(\omega)) \triangleq cx + b[\xi(\omega) - x]_+ + h[x - \xi(\omega)]_+. \tag{28}$$

Then problem of minimizing expected cost is defined as

$$\underset{x \geq 0}{\textbf{minimize}} \, \mathbb{E}\left[\, \psi(x, \boldsymbol{\xi}) \,\right]. \tag{29}$$

- From Example 26, $\psi(\bullet, \xi)$ is convex for every ξ but not necessarily smooth.

Stochastic Optimization VIII

- Consequently, the minimizers of (29) are entirely captured by the generalized variational inequality GVI(Φ, X) where

$$\Phi(x) \triangleq \partial_x \mathbb{E}\left[\, \psi(x, \boldsymbol{\xi}) \,\right] \;=\; \mathbb{E}\left[\partial_x \psi(x, \boldsymbol{\xi})\right].$$

- Interchange between the subdifferential and expectation operator holds under suitable regularity conditions (cf. [Shapiro et al., 2009, Chapter 7]).
- In fact, one may observe that the expectation $\mathbb{E}\left[\partial_x \psi(x, D(\boldsymbol{\omega}))\right]$ is an integral of a set.
- We may then define the stochastic generalized VI, where Φ is an expectation of a set-valued random map.

Stochastic Optimization IX

Definition 27 (Stochastic generalized VI(X, Φ))

Let $\Phi : \mathcal{O} \rightrightarrows \mathbb{R}^n$ and $\tilde{\Phi} : \mathcal{O} \times \Omega \rightrightarrows \mathbb{R}^n$ be set-valued maps with closed and nonempty images, where $\mathcal{O}$ is an open set containing the closed set X and $\Phi(x) = \mathbb{E}\left[\tilde{\Phi}(x, \xi)\right]$.

The stochastic generalized variational inequality problem (SGVI), defined by the pair $(\tilde{\Phi}, X)$, is to find a pair $(\bar{x}, \bar{y}) \in X \times \Phi(\bar{x})$ such that

$$\bar{y}^\top (x - \bar{x}) \geq 0, \quad \forall x \in X.$$

□

- Consequently, $\bar{x}$ is a minimizer of the single-stage newsvendor problem if and only $\bar{x}$ is a solution of SGVI($\partial\tilde{\Phi}, \mathcal{X}$).

Stochastic Nash Equilibrium Problems I

- Consider an N-player noncooperative game in which the ith player is characterized by a real-valued random cost function $\tilde{\theta}_i : \mathcal{O} \times \Omega \to \mathbb{R}$, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a random variable $\boldsymbol{\xi}$, defined as $\boldsymbol{\xi} : \Omega \to \mathbb{R}^d$, and a strategy set X^i, a closed subset of $\mathcal{O}_i \subseteq \mathbb{R}^{n_i}$.
- As defined realier, $\mathcal{O}$ is an open set containing the Cartesian product $X = \prod_{i=1}^{N} X^i$, where the elements of X are the tuples of player strategies. Furthermore, we remind the reader that elements of X^i and $X^{-i} \triangleq \prod_{j \neq i} X^j$ are denoted by x^i and x^{-i}, respectively.
- Anticipating the tuple $x^{-i} \in X^{-i}$ of rivals' strategies, a (risk-neutral) player i's objective θ_i is defined as

$$\theta_i(x^i, x^{-i}) \triangleq \mathbb{E}\left[\tilde{\theta}_i(x^i, x^{-i}, \boldsymbol{\xi}) \right], \tag{30}$$

and she makes her decision by solving the risk-neutral problem:

$$\underset{x^i \in X^i}{\textbf{minimize}} \; \theta_i(x^i, x^{-i}) \triangleq \mathbb{E}\left[\tilde{\theta}_i(x^i, x^{-i}, \boldsymbol{\xi}) \right], \tag{31}$$

Stochastic Nash Equilibrium Problems II

- By defining $\tilde{\Theta} \triangleq \left(\tilde{\theta}_i\right)_{i=1}^{N}$ as the vector of player-specific random cost functions, we employ the notation $\mathcal{EG}(\tilde{\Theta}, X)$ to denote a risk-neutral Nash equilibrium problem . We now present a formal definition.

Definition 28 (Risk-neutral NEP)

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a tuple $\bar{x} \triangleq (\bar{x}^i)_{i=1}^{N} \in X$ is a *stochastic Nash equilibrium* (NE) of the game $\mathcal{EG}(\tilde{\Theta}, X)$ if

$$\bar{x}^i \in \underset{x^i \in X^i}{\mathbf{argmin}}\, \mathbb{E}\left[\tilde{\theta}_i(x^i, \bar{x}^{-i}, \boldsymbol{\xi})\right], \qquad \forall\, i \in [N].$$

$\mathcal{EG}(\tilde{\Theta}, X)$ as a *convex risk-neutral game* if X^i is a convex set for every $i \in [N]$ and for almost every ξ, $\tilde{\theta}_i(\bullet, x^{-i}, \xi)$ is a convex function for $x^{-i} \in X^{-i}$. □

Under convexity assumptions, the equilibria of the risk-neutral game can be entirely captured by a stochastic variational inequality problem.

Stochastic Nash Equilibrium Problems III

Proposition 29

Consider the N-player convex risk-averse NEP and convex risk-neutral NEP defined in Definition 28. Then the following hold.

(i) $\bar{x} \in X$ *is a risk-neutral Nash equilibrium of* $\mathcal{EG}(\tilde{\Theta}, X)$ *if and only if* $\bar{x}$ *is a solution of SVI*$(\tilde{\Phi}, X)$ *where* $\tilde{\Phi}(x, \xi)$ *is defined as*

$$\tilde{\Phi}(x, \xi) \triangleq \prod_{i=1}^{N} \tilde{\Phi}_i(x, \xi), \text{ where } \tilde{\Phi}_i(x, \xi) \triangleq \partial_{x^i} \left[\tilde{\theta}_i(x, \boldsymbol{\xi}) \right]. \tag{32}$$

Stochastic Generalized Nash Equilibrium Problems I

- In many instances, noncooperative games are characterized by the presence of coupled strategy sets, i.e. for any $i \in [N]$, we have

$$x^i \in C_i(x^{-i}), \tag{33}$$

where C_i is a set-valued map defined as $C_i : \mathbb{R}^{n_{-i}} \rightrightarrows \mathbb{R}^{n_i}$ with convex and closed images.

- Anticipating the tuple $x^{-i} \in X^{-i}$ of rivals' strategies, a (risk-neutral) player i's objective θ_i is defined as

$$\theta_i(x^i, x^{-i}) \triangleq \mathbb{E}\left[\tilde{\theta}_i(x^i, x^{-i}, \boldsymbol{\xi})\right], \tag{34}$$

and she makes her decision by solving the risk-neutral problem:

$$\begin{aligned} \textbf{minimize} \quad & \theta_i(x^i, x^{-i}) \triangleq \mathbb{E}\left[\tilde{\theta}_i(x^i, x^{-i}, \boldsymbol{\xi})\right] \\ \textbf{subject to} \quad & x^i \in C_i(x^{-i}). \end{aligned} \tag{35}$$

Stochastic Generalized Nash Equilibrium Problems II

- By defining $\tilde{\Theta} \triangleq \left(\tilde{\theta}_i\right)_{i=1}^N$ as the vector of player-specific random cost functions and $\mathcal{C}$ as the collection of set-valued maps $C_1, \cdots, C_N$, we employ the notation $\mathcal{EG}(\tilde{\Theta}, \mathcal{C})$ to denote a risk-neutral generalized Nash equilibrium problem . We now present a formal definition.

Definition 30 (Risk-neutral GNEP)

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a tuple $\bar{x} \triangleq \left(\bar{x}^i\right)_{i=1}^N \in X$ is a *stochastic generalized Nash equilibrium* (NE) of the game $\mathcal{EG}(\tilde{\Theta}, \mathcal{C})$ if

$$\bar{x}^i \in \underset{x^i \in C_i(x^{-i})}{\textbf{argmin}} \; \mathbb{E}\left[\tilde{\theta}_i(x^i, \bar{x}^{-i}, \boldsymbol{\xi})\right], \qquad \forall\, i \in [N].$$

$\mathcal{EG}(\tilde{\Theta}, X)$ as a *convex risk-neutral coupled-constraint game* if $C_i(x^{-i})$ is a convex set for any x^{-i} and any $i \in [N]$ and for almost every ξ, $\tilde{\theta}_i(\bullet, x^{-i}, \xi)$ is a convex function for $x^{-i} \in X^{-i}$. □

Stochastic Generalized Nash Equilibrium Problems III

- Before proceeding, we define the stochastic quasi-variational inequality problem.

Definition 31 (Stochastic quasi-variational inequality problem QVI(C, F))

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a single-valued map and $C : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map with closed, convex, and nonempty images

The stochastic quasi-variational inequality problem (SQVI), defined by the pair (C, F), is to find a vector $\bar{x}$ such that

$$F(\bar{x})^\top (x - \bar{x}) \geq 0, \quad \forall x \in C(\bar{x}).$$

□

Stochastic Generalized Nash Equilibrium Problems IV

- Under convexity assumptions, the equilibria of the risk-neutral coupled-constraint game can be entirely captured by a stochastic quasi-variational inequality problem.

Proposition 32 (Stochastic Convex GNEPs and Stochastic QVIs)

Consider the N-player convex risk-neutral GNEP defined in Definition 30. Then the following holds. $\bar{x}$ is a risk-neutral generalized Nash equilibrium of $\mathcal{E}\mathcal{G}(\tilde{\Theta}, C)$ if and only if $\bar{x}$ is a solution of SQVI(C, F) where $C(x) \triangleq \prod_{i=1}^{N} C_i(x^{-i})$, $F(x) \triangleq \mathbb{E}\left[\tilde{F}(x,\xi)\right]$, and $\tilde{F}(x,\xi)$ is defined as

$$\tilde{F}(x,\xi) \triangleq \begin{pmatrix} \nabla_{x^1}\left[\tilde{\theta}_1(x,\boldsymbol{\xi})\right] \\ \vdots \\ \nabla_{x^N}\left[\tilde{\theta}_N(x,\boldsymbol{\xi})\right] \end{pmatrix}. \tag{36}$$

Summary I

$$[\,x \text{ is an NE }] \iff [\,x \text{ solves } \mathrm{VI}(X,F)\,] \quad \text{(Smooth convex: } \mathscr{G}^{sc})$$

$$[\,x \text{ is an NE }] \iff [\,x \text{ solves } \mathrm{VI}(X,\Phi)\,] \quad \text{(Nonsm. convex: } \mathscr{G}^{ns})$$

$$[\,x \text{ is an SNE }] \iff \left[\,x \text{ solves } \mathrm{SVI}(X,\tilde{F})\right] \quad \text{(Stoch. } \mathscr{G}^{sc})$$

$$[\,x \text{ is an SNE }] \iff \left[\,x \text{ solves } \mathrm{SVI}(X,\tilde{\Phi})\right] \quad \text{(Stoch. } \mathscr{G}^{ns})$$

$$[\,x \text{ is an SGNE }] \iff \left[\,x \text{ solves } \mathrm{SQVI}(X,\tilde{F})\right] \quad \text{(Stoch. gener. } \mathscr{G}^{sc})$$

$$[\,x \text{ is an NE }] \iff \left[\,x \text{ is stat. pt of } \left\{ \min_{x\in X} h(x) \right\}\right] \quad \text{(Pot. game)}$$

Existence of solutions I

Consider the variational inequality problem VI(X, F) which requires an $x \in X$ such that

$$(y - x)^T F(x) \ \geq \ 0, \qquad \forall y \ \in \ X. \qquad (\text{VI}(X, F))$$

Then the existence of a solution can be claimed as follows.

Proposition 33 ([Facchinei and Pang, 2003])

Consider VI(X, F) and suppose $X \subseteq \mathbb{R}^n$ is closed, convex, and nonempty and $F(x)$ is a continuous mapping defined as $F : X \to \mathbb{R}^n$. Suppose one of the following holds.

(a) *Suppose X is compact.*

(b) *Suppose there exists an $x^{\text{ref}} \in X$ such that*

$$\liminf_{\|x\| \to \infty, x \in X} \ F(x)^T (x - x^{\text{ref}}) \ > \ 0. \qquad (37)$$

Then VI(X, F) admits a solution.

Existence of solutions II

When X is a cone, then this problem reduces to CP(X, F), defined as follows.

$$X \ni x \perp F(x) \in X^*. \qquad (\text{CP}(X,F))$$

Existence of solutions to CP(X, F) can be proven as follows.

Proposition 34

Consider CP(X, F) and suppose $X \subseteq \mathbb{R}^n$ is a closed, convex, and nonempty cone and F is a continuous mapping defined as $F : X \to \mathbb{R}^n$. Suppose E is a copositive matrix on X such that (X, E) is an $\mathbf{R}_0$ pair and

$$\bigcup_{\tau > 0} SOL(X, F + \tau E) \text{ is bounded.} \qquad (38)$$

Then CP(X, F) admits a solution.

(X, E) *is an* $\mathbf{R}_0$ *pair if* $SOL(X_\infty, 0, E) = \{0\}$.

Challenge

In stochastic variational problems, existence statements are difficult to acquire:

1. Any result is directly tied to distribution function
2. Integration (wrt measure) introduces significant complexity, generally precluding a “deterministic” analysis

- Earliest formulations can be traced to [King and Rockafellar, 1993]; modeled as a generalized equation with an expectation-valued map.
- Stability statements examined by [Liu, Romisch, and Xu, 2014]
- Little available for existence analysis of such problems.

Goal: Develop sufficiency conditions that do NOT require integration

Existence analysis for stochastic games

- Integrable quadratic games
- Non-integrable quadratic games
- Generalizations

Integrable Quadratic Games I

For $i \in [N]$, associated with the ith player are random matrices $\tilde{Q}_{ij} : \Xi \to \mathbb{R}^{n_i \times n_j}$ for $j \in [N]$ and a random vector $d_i : \Xi \to \mathbb{R}^{n_i}$. Consequently, for $i \in [N]$, the ith player is characterized by the following problem with an expectation-valued quadratic objective, denoted by θ_i and defined as

$$\theta_i(x) \triangleq \mathbb{E}\left[\tilde{\theta}_i(x^i, x^i, \boldsymbol{\xi})\right], \tag{39}$$

$$\text{where } \tilde{\theta}_i(x^i, x^{-i}, \xi) \triangleq \tfrac{1}{2}(x^i)^\top \tilde{Q}_{ii}(\xi) x^i + \sum_{j \neq i} (x^j)^\top \tilde{Q}_{ij}(\xi) x^i + \tilde{d}_i(\xi)^\top x^i. \tag{40}$$

Suppose for $i \in [N]$, the ith player is faced by the following problem where θ_i is defined as (39), $\tilde{A}_i : \Xi \to \mathbb{R}^{m_i \times n_i}$, and $\tilde{b}_i : \Xi \to \mathbb{R}^{m_i}$.

$$\begin{aligned} \underset{x^i \geq 0}{\textbf{minimize}} \quad & \theta_i(x^i, x^{-i}) \\ \textbf{subject to} \quad & \mathbb{E}\left[\tilde{A}_i(\boldsymbol{\xi}) x^i - \tilde{b}_i(\boldsymbol{\xi})\right] \geq 0. \end{aligned} \tag{41}$$

Integrable Quadratic Games II

We refer to this quadratic game as $\mathcal{EG}(\tilde{Q}, \tilde{d}, \tilde{A}, \tilde{q})$. The following assumption ensures the convexity of player problems.

Assumption 1

For $i \in [N]$, suppose $\tilde{Q}_{ii}(\xi)$ is a positive semidefinite matrix for a.e. $\xi \in \Xi$. $\square$

Before proceeding, we define the stochastic counterpart of the linear complementarity problem.

Integrable Quadratic Games III

Definition 35 (Stochastic LCP)

Suppose $\tilde{M} : \Xi \to \mathbb{R}^{n \times n}$ and $\tilde{q} : \Xi \to \mathbb{R}^n$ denote a random matrix and vector respectively. Then the stochastic linear complementarity problem SLCP$(\tilde{q}, \tilde{M})$ requires an x such that

$$0 \leq x \perp \mathbb{E}\left[\tilde{M}(\xi)\right] x + \mathbb{E}\left[\tilde{q}(\xi)\right] \geq 0. \qquad (\text{SLCP}(\tilde{q}, \tilde{M}))$$

Suppose $\tilde{M}$ is a degenerate random matrix and $\tilde{q}$ is a degenerate random vector, i.e. $\tilde{M}(\xi) = M$ and $\tilde{q}(\xi) = q$ for every $\xi \in \Xi$. Then SLCP$(\tilde{q}, \tilde{M})$ reduces to LCP(q, M), defined above. □

In this section, we consider subclasses of games where the equilibrium conditions (captured by a linear complementarity problem) can be equivalently viewed as the necessary and sufficient optimality conditions of a quadratic program, i.e *integrable quadratic games*.

Integrable Quadratic Games IV

Example 36

Consider a Cournot game where the inverse demand function, a random variable denoted by $\tilde{p}\left(\sum_{i=1}^{N} x^i, \boldsymbol{\xi}\right)$, defined as $\tilde{p}\left(\sum_{i=1}^{N} x^i, \xi\right) = a(\xi) - r(\xi)\sum_{i=1}^{N} x^i$, where $a : \Xi \to \mathbb{R}_{++}$ and $r : \Xi \to \mathbb{R}_{++}$. For $i \in [N]$, the ith player is characterized by a random cost function $\frac{1}{2}c_i(\boldsymbol{\xi})(x^i)^2 + d_i(\boldsymbol{\xi})x^i$ for $i \in [N]$ where $c_i : \Xi \to \mathbb{R}_+$ and $d_i : \Xi \to \mathbb{R}$. Consequently, the ith player's objective is defined as

$$\theta_i(x^i, x^{-i}) \triangleq \mathbb{E}\left[\frac{1}{2}c_i(\boldsymbol{\xi})(x^i)^2 + d_i(\boldsymbol{\xi})x^i - \tilde{p}\left(\sum_{i=1}^{N} x^i, \boldsymbol{\xi}\right) x^i\right] \tag{42}$$

for $i \in [N]$.

(a) $\bar{x} = \{x^1, \cdots, x^N\}$ is a Nash-Cournot equilibrium if and only if $\bar{x}$ is a solution to SLCP($\tilde{q}$, $\tilde{M}$) where

$$\tilde{q}(\boldsymbol{\xi}) \triangleq \begin{pmatrix} d_1(\boldsymbol{\xi}) - a(\boldsymbol{\xi}) \\ \vdots \\ d_N(\boldsymbol{\xi}) - a(\boldsymbol{\xi}) \end{pmatrix} \text{ and } \tilde{M}(\boldsymbol{\xi}) \triangleq \begin{pmatrix} c_1(\boldsymbol{\xi}) + 2r(\boldsymbol{\xi}) & \cdots & r(\boldsymbol{\xi}) \\ \vdots & \ddots & \vdots \\ r(\boldsymbol{\xi}) & \cdots & c_N(\boldsymbol{\xi}) + 2r(\boldsymbol{\xi}) \end{pmatrix}.$$

Integrable Quadratic Games V

Example 37

(b) Since $\tilde{M}(\xi)$ is a symmetric positive semidefinite matrix for every $\xi \in \Xi$, $\bar{x}$ is a Nash-Cournot equilibrium if and only if $\bar{x}$ is a solution to the following convex quadratic program, where I denotes the identity matrix and $\mathbf{1}$ represents the column of ones.

$$\underset{x \geq 0}{\text{minimize}} \quad \mathbb{E}\left[\left(\sum_{i=1}^{N} \tfrac{1}{2} c_i(\boldsymbol{\xi})(x^i)^2 + (d_i(\boldsymbol{\xi}) - a(\boldsymbol{\xi}))\, x^i \right) + r(\boldsymbol{\xi})\, x^\top \left(I + \mathbf{1}\mathbf{1}^\top \right) x \right],$$

We may now generalize the previous example to derive such a relationship for the stochastic quadratic game given by $\mathcal{EG}(\tilde{Q}, \tilde{A}, \tilde{d}, \tilde{b})$. This necessitates defining the random vector $\tilde{q}(\boldsymbol{\xi})$ and the random matrix $\tilde{M}(\boldsymbol{\xi})$ as

$$\tilde{q}(\boldsymbol{\xi}) \triangleq \begin{pmatrix} \tilde{d}(\boldsymbol{\xi}) \\ -\tilde{b}(\boldsymbol{\xi}) \end{pmatrix}, \text{ where } \tilde{d}(\boldsymbol{\xi}) \triangleq \begin{pmatrix} \tilde{d}_1(\boldsymbol{\xi}) \\ \vdots \\ \tilde{d}_N(\boldsymbol{\xi}) \end{pmatrix}, \tilde{b}(\boldsymbol{\xi}) \triangleq \begin{pmatrix} \tilde{b}_1(\boldsymbol{\xi}) \\ \vdots \\ \tilde{b}_N(\boldsymbol{\xi}) \end{pmatrix} \qquad (43)$$

Integrable Quadratic Games VI

and

$$\tilde{M}(\xi) = \begin{bmatrix} \tilde{Q}(\xi) & -\tilde{A}(\xi)^\top \\ \tilde{A}(\xi) & \end{bmatrix}, \tag{44}$$

where

$$\tilde{Q}(\xi) \triangleq \begin{pmatrix} \tilde{Q}_{11}(\xi) & \cdots & \tilde{Q}_{1N}(\xi) \\ \vdots & \ddots & \vdots \\ \tilde{Q}_{N1}(\xi) & \cdots & \tilde{Q}_{NN}(\xi) \end{pmatrix} \text{ and } \tilde{A}(\xi) \triangleq \begin{pmatrix} \tilde{A}_1(\xi) & & \\ & \ddots & \\ & & \tilde{A}_N(\xi) \end{pmatrix}. \tag{45}$$

We observe that the random matrix $\tilde{M}(\xi)$ has a block skew-symmetric structure, where $\tilde{Q}(\xi)$ is matrix-valued random variable taking symmetric and positive semidefinite values in $\mathbb{R}^{n \times n}$. Consequently, it can be shown that the

Integrable Quadratic Games VII

equilibrium conditions represent the optimality conditions of the following quadratic program with expectation-valued objectives and constraints where $\tilde{Q} : \Xi \to \mathbb{R}^{n \times n}$, $\tilde{A} : \Xi \to \mathbb{R}^{m \times n}$, $\tilde{d} : \Xi \to \mathbb{R}^n$, and $\tilde{b} : \Xi \to \mathbb{R}^m$, where $m \triangleq \sum_{i=1}^{N} m_i$ and $n \triangleq \sum_{i=1}^{N} n_i$.

$$\begin{aligned} \underset{x \geq 0}{\textbf{minimize}} \quad & \mathbb{E}\left[\tfrac{1}{2} x^\top \tilde{Q}(\xi)\, x + \tilde{d}(\xi)^\top x \right] \\ \textbf{subject to} \quad & \mathbb{E}\left[A(\xi)x - b(\xi) \right] \geq 0. \end{aligned} \qquad \text{(Stoch-QP)}$$

Integrable Quadratic Games VIII

Proposition 38

[Integrable Quadratic Games] Consider the N-player game given by $\mathcal{EG}(\tilde{Q}, \tilde{A}, \tilde{q}, \tilde{d})$ in which the ith player solves (41)*. Then the following hold.*

(a) Suppose Assumption 1 holds.

$$\left[\bar{x} \text{ is an NE of } \mathcal{EG}(\tilde{Q}, \tilde{A}, \tilde{q}, \tilde{d})\right] \iff \left[(\bar{x}, \bar{\lambda}) \text{ solves SLCP}(\tilde{q}, \tilde{M}).\right] \quad (46)$$

(b) Suppose Assumption 1 holds and $\tilde{Q}(\xi)$ is a symmetric positive semidefinite matrix for almost every $\xi \in \Xi$.

$$\left[\bar{x} \text{ is an NE of } \mathcal{EG}(\tilde{Q}, \tilde{A}, \tilde{q}, \tilde{d})\right] \iff [\bar{x} \text{ solves (Stoch-QP).}] \quad (47)$$

□

Integrable Quadratic Games IX

Definition 39 (Stochastic generalizations of positive definiteness)

Consider the random matrix $\tilde{M} : \Xi \to \mathbb{R}^{n \times n}$. Then the following hold.

(a) $\tilde{M}(\boldsymbol{\xi})$ is said to be a random positive semidefinite matrix if $\tilde{M}(\xi) \succeq 0$ for a.e. $\xi \in \Xi$; we characterize the positive semidefiniteness of $\tilde{M}(\boldsymbol{\xi})$ as $\tilde{M}(\boldsymbol{\xi}) \in \mathbf{P^{sd}}$.

(b) $\tilde{M}(\boldsymbol{\xi})$ is said to be a random positive definite matrix if $\tilde{M}(\boldsymbol{\xi})$ is a random positive semidefinite matrix and

$$\tilde{M}(\xi) \succ 0 \text{ for } \xi \in \Xi_c, \text{ where } \Xi_c \triangleq \{ \xi(\omega) \mid \omega \in \Omega_c \subseteq \Omega \text{ where } \mu[\Omega_c] > 0 \}.$$

We characterize positive definiteness of $\tilde{M}(\boldsymbol{\xi}) \in \mathbf{P}^{\mathbf{d}}_{\Xi_c}$. Note that if $\Xi_c = \Xi$, then $\tilde{M}(\xi) \succ 0$ for a.e. $\xi \in \Xi$ and we classify such a random matrix as $\tilde{M}(\boldsymbol{\xi}) \in \mathbf{P^d}$. □

Integrable Quadratic Games X

Lemma 40

Consider the random matrix $\tilde{M} : \Xi \to \mathbb{R}^{n\times n}$.

(a) Suppose $\tilde{M}(\boldsymbol{\xi}) \in \mathbf{P^{sd}}$. *Then* $\mathbb{E}\left[\tilde{M}(\boldsymbol{\xi})\right] \succeq 0$, *i.e.*

$$\left[\tilde{M}(\xi) \succeq 0 \text{ for a.e. } \xi \in \Xi\right] \implies \mathbb{E}\left[x^\top M(\boldsymbol{\xi})\, x\right] \geq 0, \quad \forall\, x \in \mathbb{R}^n.$$

(b) Suppose $\tilde{M}(\boldsymbol{\xi}) \in \mathbf{P}^{\mathbf{d}}_{\Xi_c}$. *Then* $\mathbb{E}\left[\tilde{M}(\boldsymbol{\xi})\right] \succ 0$, *i.e.*

$$\left[\tilde{M}(\xi) \succ 0 \text{ for } \xi \in \Xi_c\right] \implies \mathbb{E}\left[x^\top M(\boldsymbol{\xi})\, x\right] > 0, \quad \forall\, 0 \neq x \in \mathbb{R}^n,$$

where Ξ_c *is defined in* (39). □

Integrable Quadratic Games XI

Proposition 41 (Sufficiency conditions: Integrable Quadratic Games)

Consider the quadratic game $\mathcal{EG}(\tilde{Q}, \tilde{A}, \tilde{d}, \tilde{b})$*. Suppose the random matrix* $\tilde{Q}(\boldsymbol{\xi})$ *satisfies* $\tilde{Q}(\boldsymbol{\xi}) \in \mathbf{P^{sd}}$ *and* $\tilde{\theta}(x, \xi) \triangleq \frac{1}{2} x^\top \tilde{Q}(\xi)\, x + \tilde{d}(\xi)^\top x$*, and* X *is assumed to be a nonempty polyhedron defined as* $X \triangleq \{ x \mid \mathbb{E}[A(\boldsymbol{\xi})x - b(\boldsymbol{\xi})] \geq 0, x \geq 0 \}$*. Then* $\mathcal{EG}(\tilde{Q}, \tilde{A}, \tilde{d}, \tilde{b})$ *admits a Nash equilibrium if one of the following hold.*

(a) Suppose X *is bounded.*
(b) Suppose there exists a deterministic scalar b such that the function $\tilde{\theta}(x, \xi) \geq b$ *for any* $x \in X$ *and almost every* $\xi \in \Xi$*.*
(c) Suppose there exists a vector $x^{\mathrm{ref}} \in X$ *such that*

$$\liminf_{x \in X, \|x\| \to \infty} \left(\tilde{Q}(\xi)\, x^{\mathrm{ref}} + \tilde{d}(\xi) \right)^\top \left(x - x^{\mathrm{ref}} \right) > 0 \text{ holds a.s.} \tag{48}$$

and there exists a nonnegative integrable function $u : \Xi \to \mathbb{R}$ *such that*

$$\left(\tilde{Q}(\xi)\, x^{\mathrm{ref}} + \tilde{d}(\xi) \right)^\top \left(x - x^{\mathrm{ref}} \right) \geq -u(\xi) \text{ holds a.s. for any } x. \tag{49}$$

Non-integrable Quadratic Games I

$$[\, x \text{ is an NE }] \iff \left[x \text{ solves SLCP}(\tilde{q}, \tilde{M}) \right].$$

In other words, we focus on developing integration-free conditions for solvability of SLCP($\tilde{q}, \tilde{M}$).

Theorem 42

Suppose $\tilde{M}(\xi)$ and $\tilde{q}(\xi)$ represent a random matrix and random vector, respectively. Suppose $\tilde{M}(\xi) \in \mathbf{P^{sd}}$. If SLCP($\tilde{q}, \tilde{M}$) is feasible, then it is solvable.

In other words,

$$\left[\exists x \geq 0, \mathbb{E}\left[\tilde{M}(\xi)x + \tilde{q}(\xi) \right] \geq 0 \right] \implies \left[\text{SLCP}(\tilde{q}, \tilde{M}) \text{ is solvable .} \right] \quad (50)$$

Non-integrable Quadratic Games II

Lemma 43

Suppose $\tilde{M}(\xi) \in \mathbf{P}^{\mathbf{d}}_{\Xi_c}$. Then there exists a vector z such that

$$\mathbb{E}\left[\tilde{M}(\xi)z\right] > 0, \qquad z > 0. \tag{51}$$

Proof sketch. Leverage Ville's theorem of the alternative + positive definiteness.

Recall that a matrix M is said to be an **S**-matrix (where **S** stands for Stiemke) if there exists a z such that

$$Mz > 0, \quad z > 0. \tag{52}$$

In fact, the above claim holds if and only if

$$Mz > 0, \quad z \geq 0. \tag{53}$$

Non-integrable Quadratic Games III

Naturally, (53) is implied by (52). The reverse claim holds by observing that since $M(z + \lambda \mathbf{1})$ for λ sufficiently small since $Mz > 0$. Consequently, we have discovered a vector $z + \lambda \mathbf{1} > 0$ satisfying $M(z + \lambda \mathbf{1}) > 0$ since $z \geq 0$. We now introduce the definition of a random **S** matrix.

Definition 44 (Random S matrix)

The random matrix, defined as $\tilde{M} : \Xi \to \mathbb{R}^{n \times n}$, satisfies $\tilde{M}(\boldsymbol{\xi}) \in \mathbf{S}_{\Xi_c}$ if there exists a deterministic vector $z > 0$ such that

$$\tilde{M}(\xi)\, z \geq 0, \forall \xi \in \Xi \text{ and } \tilde{M}(\xi)\, z > 0, \forall \xi \in \Xi_c,$$

where $\Xi_c \triangleq \{\, \xi(\omega) \mid \omega \in \Omega_c \subseteq \Omega \text{ where } \mu\, [\Omega_c] > 0 \,\}$. □

Non-integrable Quadratic Games IV

Lemma 45 (Positive definiteness $\implies$ S)

Suppose $\tilde{M}(\xi)$ represents a random matrix. Then

$$\tilde{M}(\xi) \in \mathbf{P}^{\mathbf{d}}_{\Xi_c} \quad \implies \quad \tilde{M}(\xi) \in \mathbf{S}_{\Xi_c}.$$

Proposition 46

Suppose $\tilde{M}(\xi)$ and $\tilde{q}(\xi)$ represent a random matrix and random vector, respectively. Suppose $\tilde{M}(\xi) \in \mathbf{S}_{\Xi_c}$. Then SLCP$(\tilde{q}, \tilde{M})$ is feasible for all $\tilde{q}(\xi)$.

Consequently, we have an *integration-free* condition for claiming that SLCP$(\tilde{q}, \tilde{M})$ is feasible. Consequently, the following integration-free sufficiency condition holds.

$$\left[\tilde{M}(\xi) \in \mathbf{P}^{\mathbf{sd}} \cap \mathbf{S}_{\Xi_c} \right] \implies \left[\text{SLCP}(\tilde{q}, \tilde{M}) \text{ is solvable } \right]. \qquad (54)$$

Non-integrable Quadratic Games V

Proposition 47 (Uniqueness of SLCP($\tilde{q}, \tilde{M}$))

Suppose $\tilde{M}(\xi)$ and $\tilde{q}(\xi)$ represent a random matrix and random vector, respectively. Suppose $\tilde{M}(\xi) \in \mathbf{P}^{\mathbf{d}}_{\Xi_c}$. Then SLCP($\tilde{q}, \tilde{M}$) admits a unique solution for all $\tilde{q}(\xi)$.

Non-integrable Quadratic Games VI

We now introduce the stochastic counterpart of a copositive matrix. Recall that $M \in \mathbb{R}^{n\times n}$ is said to be a copositive matrix, if $x^\top M x \geq 0$ for any $x \in \mathbb{R}^n_+$.

Definition 48 (Stochastic generalizations of Copositivity)

Consider the random matrix $\tilde{M} : \Xi \to \mathbb{R}^{n\times n}$ and a cone $X \subseteq \mathbb{R}^n$. Then the following hold.

(a) $\tilde{M}(\boldsymbol{\xi})$ is said to be a random copositive matrix on a cone X if for a.e. $\xi \in \Xi$, we have that

$$x^\top \tilde{M}(\xi)\, x \;\geq\; 0 \text{ for all } x \;\in\; X;$$

we denote such a random matrix as $\tilde{M}(\boldsymbol{\xi}) \in \mathbf{CP}(X)$.
(b) $\tilde{M}(\boldsymbol{\xi})$ is said to be a random strictly copositive matrix if $\tilde{M}(\boldsymbol{\xi})$ is a random copositive matrix on X and for any $\xi \in \Xi_c$, we have

$$x^\top \tilde{M}(\xi)\, x \;>\; 0 \text{ for any } x \;\in\; X,\; x \neq 0$$

where $\Xi_c \triangleq \{\, \xi(\omega) \mid \omega \in \Omega_c \subseteq \Omega \text{ and } \mu\,[\Omega_c] > 0 \,\}$. We denote such a random matrix as $\tilde{M}(\boldsymbol{\xi}) \in \mathbf{CP}^{\mathbf{st}}_{\Xi_c}(X)$.
(c) If $\Xi_c = \Xi$, then for a.e. $\xi \in \Xi$, $x^\top \tilde{M}(\xi)x > 0$ for any $0 \neq x \in X$ and we refer to this random matrix as $\tilde{M}(\boldsymbol{\xi}) \in \mathbf{CP}^{\mathbf{st}}(X)$.
(d) If $X \equiv \mathbb{R}^n_+$, then the random matrix classes in (a) – (c) reduce to $\mathbf{CP}$, $\mathbf{CP}^{\mathbf{st}}_{\Xi_c}$, and $\mathbf{CP}^{\mathbf{st}}$, respectively. □

Non-integrable Quadratic Games VII

Our first result provides a sufficiency condition for existence of a solution of SLCP$(\tilde{q}, \tilde{M})$ by leveraging strict copositivity of the random matrix $\tilde{M}(\xi)$.

Theorem 49 (Solvability under strict copositivity)

Let $\tilde{M}(\xi)$ represent a given random matrix. If $\tilde{M}(\xi) \in \mathbf{CP}^{\mathbf{st}}_{\Xi_c}$ respectively. Then SLCP$(\tilde{q}, \tilde{M})$ admits a solution for any random vector $\tilde{q}(\xi)$.

One might mistakenly assume that SLCP$(\tilde{q}, \tilde{M})$ need not be solvable if $\mathbb{E}\left[\tilde{M}(\xi)\right]$ is merely copositive. However, this is not true. There are indeed instances when $\mathbb{E}\left[\tilde{M}(\xi)\right]$ is copositive and solvability holds. The following example provides precisely such an instance.

Non-integrable Quadratic Games VIII

Example 50 (Existence under copositivity)

(a) Consider a noncooperative game in which for $i \in [N]$, the ith player solves (QP$_i(x^{-i})$), defined as

$$\underset{x_i \geq 0}{\textbf{minimize}} \quad \mathbb{E}\left[\tfrac{1}{2}\boldsymbol{\xi} x_i^2 + \sum_{j \neq i} \boldsymbol{\xi}\, x_i x_j + \tilde{d}_i(\boldsymbol{\xi})\, x_i\right]. \qquad (\text{QP}_i(x^{-i}))$$

(b) Suppose $\boldsymbol{\xi}$ has mean zero. Then the necessary and sufficient equilibrium conditions are given by SLCP($\tilde{q}$, $\tilde{M}$) where $\tilde{M}(\xi)$ and $\tilde{q}(\xi)$ are defined as

$$\tilde{M}(\xi) = \begin{pmatrix} \xi & \cdots & \xi \\ \vdots & \ddots & \vdots \\ \xi & \cdots & \xi \end{pmatrix} \text{ and } \tilde{q}(\xi) \triangleq \begin{pmatrix} d_1(\xi) \\ \vdots \\ d_N(\xi) \end{pmatrix}, \text{ respectively.}$$

(c) Consequently, $\mathbb{E}\left[\tilde{M}(\boldsymbol{\xi})\right] = (0)_{N \times N}$. Consequently, SLCP($\tilde{q}$, $\tilde{M}$) has a solution if $\mathbb{E}\left[\tilde{q}(\boldsymbol{\xi})\right] \geq 0$. □

Non-integrable Quadratic Games IX

Theorem 51 (Solvability under copositivity)

*Let $\tilde{M}(\boldsymbol{\xi})$ and $\tilde{q}(\boldsymbol{\xi})$ represent a random matrix and a random vector, respectively. Let $\tilde{M}(\boldsymbol{\xi}) \in$ **CP** and the following implication holds for almost every $\xi \in \Xi$ for a deterministic vector $v \in \mathbb{R}^n$.*

$$\left[v \geq 0,\ \tilde{M}(\xi)\, v \geq 0,\ v^\top \tilde{M}(\xi)\, v = 0 \right] \quad \Longrightarrow \quad \left[v^\top \tilde{q}(\xi) \geq 0. \right] \qquad (55)$$

Then SLCP$(\tilde{q}, \tilde{M})$ admits a solution.

Existence of stochastic Nash equilibria I

We begin by considering the risk-neutral convex game defined by compact strategy sets and denoted by $\mathcal{EG}(\tilde{\Theta}, \Xi)$ and defining the player i's *regularized risk-neutral best response map* $R^i_{\theta,\mathcal{E}} : X^i \to X^i$ where

$$\mathcal{R}^i_{\theta_i,\mathcal{E}}(x) \triangleq \operatorname*{\mathbf{argmin}}_{y^i \in X^i} \left[\mathbb{E}\left[\tilde{\theta}_i(y^i, x^{-i}, \boldsymbol{\xi})\right] + \tfrac{1}{2} \| y^i - x^i \|^2 \right]. \tag{56}$$

The player-specific best-response map $R^i_{\theta,\mathcal{E}}$ is well-defined when X^i is nonempty, closed, and convex and $\mathbb{E}\left[\tilde{\theta}_i(\bullet, x^{-i}, \boldsymbol{\xi})\right]$ is convex and therefore continuous. The best-response map $\mathcal{R}_{\theta,X}$ is defined as

$$\mathcal{R}_{\theta,\mathcal{E}}(x) \triangleq \begin{pmatrix} \mathcal{R}^1_{\theta_1,\mathcal{E}}(x) \\ \vdots \\ \mathcal{R}^N_{\theta_N,\mathcal{E}}(x) \end{pmatrix}, \tag{57}$$

where $\mathcal{R}^i_{\theta_i,\mathcal{E}}(x)$ is defined as (56).

Existence of stochastic Nash equilibria II

Proposition 52

Consider the game $\mathcal{EG}(X, \tilde{\Theta})$ where for $i \in [1, N]$, X^i is a closed and convex subset of $\mathbb{R}^{n_i}$, $\theta_i(\bullet, x^{-i})$ is a convex function for any $x^{-i} \in X^{-i}$, and $\sum_{i=1}^{N} n_i = n$. Then $\bar{x}$ is a fixed point of $\mathcal{R}_{\theta,\mathcal{E}}$, defined as (57)*, if and only if $\bar{x}$ is an SNE of $\mathcal{EG}(X, \tilde{\Theta})$.*

We now provide conditions under which existence of a *stochastic* Nash equilibrium exists. The first of these results is a simple application of Brouwer's fixed-point theorem that does not necessitate smoothness of θ_i.

Existence of stochastic Nash equilibria III

Theorem 53 (Existence of NE under compactness)

For $i \in [1, N]$, let $X^i \subseteq \mathbb{R}^{n_i}$ be a compact and convex set and let $\theta_i : \prod_{i=1}^{N} X^i \to \mathbb{R}$ be such that $\theta_i(x^i, x^{-i}) \triangleq \mathbb{E}\left[\tilde{\theta}_i(x^i, x^{-i}, \boldsymbol{\xi})\right]$, where for almost every $\xi \in \Xi$, $\tilde{\theta}_i(\bullet, x^{-i}, \xi)$ is convex on X^i for all x^{-i} in X^{-i}. Then an SNE of the game $\mathcal{EG}(\tilde{\Theta}, X)$ exists. □

- When boundedness of X is weakened, existence claims are less easily made.
- We initiate our discussion with an example of a stochastic linear complementarity problem. In this particular instance, we observe that barring a pathological case, deriving an existence claim is not easy.

Existence of stochastic Nash equilibria IV

Example 54

Consider the following stochastic linear complementarity problem CP($\tilde{F}, \mathbb{R}^2_+$) where $0 \leq x \perp \mathbb{E}\left[\tilde{F}(x, \boldsymbol{\xi})\right] \geq 0$, where $\tilde{F}(x,\xi) \triangleq \begin{pmatrix} 2+\xi & 0.1 \\ 0.1 & 2+\xi \end{pmatrix} x - \begin{pmatrix} 2 \\ 4 \end{pmatrix}$. CP($\tilde{F}, \mathbb{R}^2_+$) is equivalent to VI($\tilde{F}, \mathbb{R}^2_+$). Consider two cases that pertain to either when the expectation is available in closed-form (a); or not (b).
(a) Expectation $\mathbb{E}[\bullet]$ *available in closed form.* Suppose in this instance, ξ is a random variable that takes values ξ^1 of ξ^2, given by the following:

$$\xi^1 = 1.1 \text{ or } \xi^2 = -0.9 \text{ with probability } 0.5.$$

Consequently, the stochastic variational inequality problem can be expressed as

$$0 \leq x \perp \begin{pmatrix} 2.1 & 0.1 \\ 0.1 & 2.1 \end{pmatrix} x - \begin{pmatrix} 2 \\ 4 \end{pmatrix} \geq 0.$$

In fact, this problem is a strongly monotone linear complementarity problem and admits a unique solution given by $x^* \approx (1.863, 0.864)$. More generally, even if one cannot compute a solution, a common approach lies in examining coercivity properties of the map F; For instance in this case, VI($F, \mathbb{R}^2_+$) is solvable since there exists an $x^{\rm ref} \equiv (0,0) \in \mathbb{R}^+_2$ such that (cf. [Facchinei and Pang, 2003, Ch. 2])

$$\liminf_{\|x\| \to \infty, x \in \mathbb{R}^+_2} F(x)^\top (x - x^{\rm ref}) = \infty.$$

Existence of stochastic Nash equilibria V

Example 55

(b) Expectation $\mathbb{E}[\bullet]$ *unavailable in closed-form.* However, in many practical settings, closed-form expressions of the expected value map are unavailable. Two possible avenues are available:

(i) If X is compact, under continuity of the expected value map, VI(F, X) can be claimed to be solvable. In this case, since X is a cone, this avenue cannot be employed.

(ii) If there exists a single $x \in X$ that solves VI(X, $F(\bullet, \xi)$) for almost every $\xi \in \Xi$, VI(X, F) is solvable. This appears to be possible only for pathological examples; in this case, there does not exist a single x that solves the scenario-based VI(X, $\tilde{F}(\bullet, \xi)$) for every $\xi \in \xi$. Specifically, $F(\bullet, \xi)$ is a strongly monotone map on $\mathbb{R}^2_+$ for $\xi = \xi^1, \xi^2$. Consequently, VI($F(\bullet, \xi^1), \mathbb{R}^2_+$) and VI($F(\bullet, \xi^2), \mathbb{R}^2_+$) each have unique solutions given by $x(\xi^1) \approx (1.27, 0.60)$ and $x(\xi^2) = (3.5, 1.5)$, respectively and since $x(\xi^1) \neq x(\xi^2)$, avenue (ii) cannot be traversed.

We intend to use the following coercivity result to derive an existence claim for the stochastic Nash equilibrium problem. Such coercivity results appear to have been first provided by Moré (1974); our statement originates from [Facchinei and Pang, 2003].

Existence of stochastic Nash equilibria VI

Theorem 56 (Existence of solution to VI(X, F))

Consider the variational inequality problem VI(F, X) where $F : X \to \mathbb{R}^n$ is a single-valued and continuous map and $X \subseteq \mathbb{R}^n$ is a closed and convex set. Suppose one of the following holds.
(a) Suppose there exists an $x^{\mathrm{ref}} \in X$ such that

$$\liminf_{x \in X,\, \|x\| \to \infty} F(x)^\top (x - x^{\mathrm{ref}}) > 0. \tag{58}$$

(b) Suppose F is a monotone map on X and there exists an $x^{\mathrm{ref}} \in X$ such that

$$\liminf_{x \in X,\, \|x\| \to \infty} F(x^{\mathrm{ref}})^\top (x - x^{\mathrm{ref}}) > 0. \tag{59}$$

Then VI(F, X) admits a solution. □

Existence of stochastic Nash equilibria VII

Theorem 57 (Solvability of SVI($\tilde{F}, X$))

Consider the stochastic variational inequality problem SVI($\tilde{F}, X$) where $\tilde{F} : X \times \xi \to \mathbb{R}^n$ is a single-valued and continuous map, $F_i(x) \triangleq \mathbb{E}[F_i(x, \xi)]$ for $i \in [1, N]$ and $X \subseteq \mathbb{R}^n$ is a closed and convex set. Suppose there exists an $x^{\mathrm{ref}} \in X$ such that

$$\liminf_{x \in X,\, \|x\| \to \infty} \tilde{F}(x, \xi)^\top (x - x^{\mathrm{ref}}) > 0, \text{ almost surely} \tag{60}$$

and there exists a nonnegative integrable function $u : \Xi \to \mathbb{R}$ such that

$$\tilde{F}(x, \xi)^\top (x - x^{\mathrm{ref}}) \geq -u(\xi), \text{ almost surely for any } x. \tag{61}$$

Then SVI($\tilde{F}, X$) admits a solution. □

Existence of stochastic Nash equilibria VIII

Theorem 58 (Solvability of SVI($\tilde{F}, X$) under monotonicity)

Consider the stochastic variational inequality problem SVI($\tilde{F}, X$) where $\tilde{F} : X \times \xi \to \mathbb{R}^n$ is a single-valued and continuous map, $F_i(x) \triangleq \mathbb{E}[F_i(x, \xi)]$ for $i \in [1, N]$ and $X \subseteq \mathbb{R}^n$ is a closed and convex set. Suppose F is a monotone map on the set X and there exists an $x^{\rm ref} \in X$ such that

$$\liminf_{x \in X, \, \|x\| \to \infty} \tilde{F}(x^{\rm ref}, \xi)^\top (x - x^{\rm ref}) > 0, \text{ almost surely} \tag{62}$$

and there exists a nonnegative integrable function $u : \Xi \to \mathbb{R}$ such that

$$\tilde{F}(x^{\rm ref}, \xi)^\top (x - x^{\rm ref}) \geq -u(\xi), \tag{63}$$

holds almost surely for any x. Then SVI($\tilde{F}, X$) admits a solution. □

- When contending with NEPs, X displays a Cartesian structure, i.e. $X = \prod_{i=1}^N X_i$.
- This allows for weaker claims of existence.

Existence of stochastic Nash equilibria IX

Theorem 59 (Existence of Stochastic NE)

For $i \in [1, N]$, let $X^i \subseteq \mathbb{R}^{n_i}$ be a closed and convex set contained in the open set $\mathcal{O}^i$ and let $\theta_i : \prod_{i=1}^{N} \mathcal{O}^i \to \mathbb{R}$ be such that $\theta_i(x^i, x^{-i}) \triangleq \mathbb{E}\left[\tilde{\theta}_i(x^i, x^{-i}, \boldsymbol{\xi})\right]$, where for every $\xi \in \xi$, $\tilde{\theta}_i(\bullet, x^{-i}, \omega)$ is convex and C^1 on X^i for all x^{-i} in X^{-i}. Suppose $\tilde{F}(\bullet, \xi)$ is defined as $\tilde{F}(x, \xi) \triangleq \left(\nabla_{x^i} \tilde{\theta}_i(x, \xi)\right)_{i=1}^{N}$. Suppose there exists an $x^{\text{ref}} \in X$ and an index $\upsilon \in [1, N]$ such that

$$\liminf_{x^\upsilon \in X^\upsilon,\ \|x^\upsilon\| \to \infty} \nabla_{x^\upsilon} \tilde{\theta}_\upsilon(x, \xi)^\top (x^\upsilon - x^{\text{ref},\upsilon}) > 0 \tag{64}$$

and there exists a nonnegative integrable function $u : \xi \to \mathbb{R}$ such that

$$\nabla_{x^\upsilon} \tilde{\theta}_\upsilon(x, \xi)^\top (x^\upsilon - x^{\text{ref},\upsilon}) \geq -u(\xi), \text{ almost surely for any } x. \tag{65}$$

Then the game $\mathcal{EG}(\tilde{\Theta}, X)$ admits a Nash equilibrium. □

Nonsmooth convex games I

Recall that $\{\bar{x}^1, \ldots, \bar{x}^N\}$ is a Nash equilibrium of $\mathcal{EG}(\tilde{\Theta}, X)$ if and only if $\bar{x} \triangleq \{\bar{x}^1, \ldots, \bar{x}^N\}$ is a solution of the variational inequality problem $\mathrm{VI}(\Phi, X)$, where

$$\Phi(x) \triangleq \prod_{i=1}^{N} \partial_{x^i} \mathbb{E}\left[\tilde{\theta}_i(x^i, x^{-i}, \boldsymbol{\xi})\right] \tag{66}$$

and $\partial_{x^i} \tilde{\theta}_i(x^i, x^{-i}, \xi)$ denotes the subdifferential of the ith player's (convex) objective at $x = (x^i, x^{-i})$. We initiate our discussion by providing some properties for Φ.

Nonsmooth convex games II

Proposition 60

Suppose Φ *is defined as* (66)*. Then the following hold.*

(i) Φ *can be expressed as*

$$\Phi(x) = \mathbb{E}\left[\tilde{\Phi}(x, \boldsymbol{\xi})\right], \text{ where } \Phi(x, \xi) = \prod_{i=1}^{N} \partial_{x^i} \tilde{\theta}_i(x^i, x^{-i}, \xi).$$

(ii) Φ *is an upper semicontinuous map with closed, convex, and compact images.*

(iii) $\tilde{\Phi}(x, \xi)$ *has nonempty and closed images for every* x *and every* $\xi \in \Xi$*.*

(iv) $\tilde{\Phi}(\bullet, \xi)$ *is upper semicontinuous for every* $\xi \in \Xi$ *and* $\tilde{\Phi}(x, \boldsymbol{\xi})$ *is integrably bounded for any* $x \in X$*.* □

Nonsmooth convex games III

Theorem 61 (Kien (2008))

Suppose X is a closed and convex set in $\mathbb{R}^n$ and let $\Phi : X \rightrightarrows \mathbb{R}^n$ be a lower semicontinuous multifunction with nonempty closed and convex images. Consider the following statements.

(a) Suppose there exists an $x^{\rm ref} \in X$ such that $L_<(X, \Phi)$ is bounded (possibly empty) where

$$L_<(\Phi, X) \triangleq \left\{ x \in X \,\middle|\, \inf_{y \in \Phi(x)} (x - x^{\rm ref})^T y < 0 \right\}. \tag{67}$$

(b) The variational inequality $VI(\Phi, X)$ admits a solution.

Then (a) implies (b). Furthermore, if Φ is a pseudomonotone mapping over X, then (a) is equivalent to (b). □

Using this condition, we proceed to develop distribution-free sufficiency conditions for the existence of solutions to SVI$(\tilde{\Phi}, X)$ when $\tilde{\Phi}$ and Φ, the

Nonsmooth convex games IV

expectation of $\tilde{\Phi}$, satisfy suitable properties. We begin by providing a representation result whereby any element of the set $\mathbb{E}\left[\tilde{\Phi}(x,\boldsymbol{\xi})\right]$ is shown to be a convex combination of a set of extremal selections of $\tilde{\Phi}(x,\xi)$.

Lemma 62

Suppose the probablity measure is nonatomic. Let $\tilde{\Phi}$ be a measurable integrably bounded set-valued map from $\mathbb{R}^n \times \xi$ to subsets of $\mathbb{R}^n$ with closed nonempty images. Then any $w \in \mathbb{E}\left[\tilde{\Phi}(x,\boldsymbol{\xi})\right]$ can be expressed as

$$w = \int_\Omega g(x,\xi)\, d\mathbb{P}(\omega),$$

where $g(x,\xi) \triangleq \sum_{k=0}^{n} \lambda_k(x)\tilde{g}_k(x,\xi)$, $\sum_{k=0}^{n} \lambda_k(x) = 1$, and for $k \in [0,n]$, $\lambda_k(x) \geq 0$ and $\tilde{g}_k(x,\xi)$ is an extremal selection of $\tilde{\Phi}(x,\xi)$. □

Nonsmooth convex games V

Proposition 63

Consider SVI($\tilde{\Phi}, X$). Suppose $\mathbb{P}$ is a nonatomic measure and Φ is an upper semicontinuous multifunction with nonempty, closed, and convex values. In addition, suppose the following hold.

(a) Suppose there exists an $x^{\text{ref}} \in X$ such that

$$\liminf_{x \in X, \|x\| \to \infty} \left(\inf_{w \in \tilde{\Phi}(x,\xi)} w^\top (x - x^{\text{ref}}) \right) > 0 \textit{ almost surely}.$$

(b) For the above x^{ref}, suppose there exists a nonnegative integrable random variable u such that for any $x \in X$, $g(x,\xi)^\top (x - x^{\text{ref}}) \geq -u(\xi)$ holds for any integrable selection $g(x,\xi)$ of $\tilde{\Phi}(x,\xi)$ in an almost sure sense.

Then SVI($\tilde{\Phi}, X$) is solvable. □

Applications I

This avenue has been utilized to provide existence guarantees for a range of equilibrium problems. A subset of references is provided next.

1. Imperfectly competitive electricity markets: [O. Daxhelet and Y. Smeers, 2001], [Hobbs, 2001], [Metzler, Hobbs, and Pang, 2002], [Hobbs and Pang, 2007], [Kannan, S., and Kim, 2013], [Gabriel, Conejo, Hobbs, and Ruiz, 2014]
2. Traffic equilibrium problems: [Ashtiani and Magnanti, 1981], [Friesz, Tobin, Smith, and Harker, 1983], [Patriksson, 1994]
3. Frictional contact problems: [Brogliato, 1999], [Stewart and Trinkle, 1997], [Stewart, 2000]
4. Option pricing [Brennan and Schwartz, 1977], [Jaillet, Lamberton, and Lapeyre, 1990], [Huang and Pang, 1998], [Pang and Huang, 2002]
5. Significant other work in obstacle problems, gas markets, etc.

Most models are natively deterministic

Stochastic complementarity problems (CP) I

Consider an N-player coupled-constraint game in which the ith player solves the following convex problem (Player$_i(x^{-i})$).

$$\begin{aligned} \underset{x^i \geq 0}{\textbf{minimize}} \quad & \mathbb{E}\left[\tilde{\theta}_i(x^i, x^{-i}, \boldsymbol{\xi})\right] \\ \textbf{subject to} \quad & \mathbb{E}\left[c_i(x^i, x^{-i}, \boldsymbol{\xi})\right] \leq 0. \quad (\lambda_i) \end{aligned} \qquad (\text{Player}_i(x^{-i}))$$

Under a suitable regularity condition, $x^* \triangleq \{x^{1,*}, \cdots, x^{N,*}\}$ is a generalized Nash equilibrium if and only if (x^*, λ^*) is a solution to the following stochastic complementarity problem.

$$\begin{aligned} 0 \leq x^i \perp \mathbb{E}\left[\nabla_{x^i}\tilde{\theta}_i(x^i, x^{-i}, \boldsymbol{\xi}) + \nabla_{x^i} c_i(x^i, x^{-i}, \boldsymbol{\xi})^\top \lambda_i\right] \geq 0, \qquad i \in [N] \\ 0 \leq \lambda^i \perp \mathbb{E}\left[c_i(x^i, x^{-i}, \boldsymbol{\xi})\right] \leq 0, \qquad i \in [N] \end{aligned} \qquad (68)$$

Stochastic complementarity problems (CP) II

- This problem can be cast more generally as the following stochastic complementarity problem.

$$K \ni z \perp \mathbb{E}\left[\tilde{H}(z,\omega)\right] \in K^*, \tag{69}$$

  where K is a cone, K^* denotes the dual cone, and $\tilde{H}(\bullet,\omega) : K \to K^*$ for every ω.
- This avenue is useful for analyzing a range of equilibrium problems, as well as analysing stochastic Nash games with stochastic constraints.

Stochastic complementarity problems (CP) III

Definition 64 (CP(K, q, M))

- Given a cone K in $\mathbb{R}^n$, an $n \times n$ matrix M and a vector $q \in \mathbb{R}^n$, CP(K, q, M) requires an $x \in K, Mx + q \in K^*$ such that $x^T(Mx + q) = 0$.
  ** $K^* \triangleq \{y : y^T d \geq 0, \ \forall d \in K\}$
- The recession cone associated with a set K (not necessairly a cone) is defined as
  $$K_\infty \triangleq \{d : \text{for some } x \in K, \{x + \tau d : \tau \geq 0\} \in K\}.$$
- The CP kernel of the pair (K, M) denoted by $\mathcal{K}(K, M)$ is given by
  $$\mathcal{K}(K, M) = SOL(K_\infty, 0, M)$$
- (K, M) is said to be an $\mathbf{R}_0$ pair if $\mathcal{K}(K, M) = \{0\}$.

Stochastic complementarity problems (CP) IV

From [Facchinei and Pang, 2003, Th. 2.5.6], when K is a closed and convex cone, (K, M) is an $\mathbf{R}_0$ pair if and only if the solutions of the CP(K, q, M) are uniformly bounded for all q belonging to a bounded set.

Definition 65

Let K be a cone in $\mathbb{R}^n$ and M be an $n \times n$ matrix. Then M is said to be

(a) *copositive on K* if $x^T Mx \geq 0, \quad \forall\, x \in K$;

(b) *strictly copositive on K* if $x^T Mx > 0, \quad \forall\, x \in K \backslash \{0\}$.

Stochastic complementarity problems (CP) V

Proposition 66 (Solvability of SCP(K, H))

Consider a stochastic complementarity problem SCP(K, H) where K is a closed convex cone in $\mathbb{R}^n$. Suppose the following hold:

(a) *For almost every $\omega \in \Omega$, there exists a strictly copositive matrix $M_\omega \in \mathbb{R}^{n \times n}$ on K such that (K, M_ω) is an $\mathbf{R}_0$ pair and the union*

$$\bigcup_{\tau > 0} SOL(K, H(.; \omega) + \tau M_\omega)$$

is bounded in an almost-sure sense;

(b) *The following property holds almost surely:*

$$\liminf_{\|x\| \to \infty, x \in K} x^T H(x; \omega) \geq -\beta, \beta > 0. \tag{70}$$

Then the stochastic complementarity problem SCP(K, H) admits a solution.

Applications I

Existence of solutions has been shown in varied settings via such avenues.

1. Stochastic Nash games with stochastic (expectation-valued) constraints (Ravat and S., 2011)
2. CVaR-based Risk-averse Nash-Cournot games (Ravat and S., 2011).
3. Imperfectly competitive power markets with random price functions and costs (Ravat and S., 2017)

Summary I

Almost-sure sufficiency statements for existence in stochastic regimes

- Integrable quadratic games
- Non-integrable quadratic games
- Stochastic Nash equilibria
- Stochastic generalized Nash equilibria

Crucial question

Well-behaved X + **Coercivity almost surely** $\overset{?}{\Longrightarrow}$ SVI$(X, \tilde{F})$ is solvable

Advantages:

- Averts the need for **integration**
- Analytically tractable

Literature I

- Gradient-Response Schemes regimes
  - Deterministic Gradient-based schemes.
    - Strongly monotone maps [Alpcan and Başar (2003, 2007); Pavel (2006), Pan and Pavel (2009)]
    - Monotone maps via iterative regularization [Yin, UVS and Mehta (2011); Kannan and UVS (2012)]
    - Distributed scheme (gradient response+consensus) for networked aggregative Nash games [Koshal, Nedić, and UVS]
    - Not "fully rational".
  - Stochastic gradient-based schemes:
    - Monotone expectation-valued maps [Koshal, Nedić and Shanbhag (2013)]
    - Non-Lipschitzian regimes via random smoothing [Yousefian, Nedić and Shanbhag (2016)]

Literature II

- Best-response schemes
  - Deterministic Best-response schemes:
    - Synchronous best-response schemes [Facchinei and Pang (2009); Scutari, Facchinei, Palomar, Song, and Pang (2013)]
    - Customized schemes in signal processing [Scutari, Palomar and Barbarossa (2008, 2009); Scutari and Palomar (2010)]
  - Stochastic Best-response schemes:
    - Inexact asynchronous BR schemes [Lei, UVS, Pang, and Sen].

Problem Statement I

Noncooperative game

$$\min_{x_i \in X_i} \theta_i(x_i, x^{-i}) \triangleq \mathbb{E}\left[\tilde{\theta}_i(x_i, x^{-i}, \xi)\right]$$

- $\mathcal{N} = \{1, \cdots, n\}$ is a group of n players, indexed by i;
- X_i is the strategy set of player i, $x = (x^1, \cdots, x^N)$ is a *strategy profile*;
- player i has an objective $\theta_i(x^i, x^{-i})$.
- $\xi : \Omega \to \mathbb{R}^m$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

- Nash equilibrium (NE) is a strategy profile $x^* = \{x_i^*\}_{i=1}^N$ such that no player can improve by unilateral deviation.
- Convexity of subproblems: X_i is a closed, compact, convex set; For any $x^{-i} \in \prod_{j \neq i} X^j$, $\theta_i(x_i, x^{-i})$ is C^1 and convex in $x_i \in X_i$.

Problem Statement II

- Existence of stochastic oracle returning a sample $\nabla_{x_i}\tilde{\theta}_i(x_i, y; \xi)$ such that the following holds a.s. for $i \in [N]$.

$$\mathbb{E}\left[\nabla_{x_i} f_i(x^i, x^{-i}) - \nabla_{x_i} \tilde{\theta}_i(x^i, x^{-i}; \xi) \mid x\right] = 0 \quad \text{(Unbiasedness)}$$

$$\mathbb{E}\left[\|\nabla_{x_i} \theta_i(x^i, x^{-i}) - \nabla_{x_i} \tilde{\theta}_i(x_i, x^{-i}; \xi)\|^2 \mid x\right] \leq \nu_i^2. \quad \text{(Bounded } 2^{nd} \text{ moments)}$$

- Suppose F is η-strongly monotone and L-Lipschitz on a closed and convex set X where

$$F(x) \triangleq \begin{pmatrix} \nabla_{x^1}\theta_1(x^1, x^{-i}) \\ \vdots \\ \nabla_{x^N}\theta_N(x^N, x^{-N}) \end{pmatrix} \text{ and } X \triangleq \prod_{i=1}^{N} X_i.$$

Gradient-Response Schemes under Strong Monotonicity of F I

Stochastic Gradient-Response Scheme

I. Initialization. For $i \in [N]$, choose $x^{i,0} \in X^i$, positive steplength sequence $\{\gamma_{i,\nu}\}$, and batch-size sequence $\{M^{i,\nu}\}$. Let $\nu := 0$.

II. General iteration. Suppose $w_\nu = (w_{i,\nu})_{i=1}^N$, where

$$w_{i,\nu} = \sum_{j=1}^{M^{i,\nu}} \frac{\left(\nabla_{x^i}\tilde{\theta}_i(x^\nu, \xi^{\nu,j}) - \nabla_{x^i}\theta_i(x^\nu)\right)}{M^{i,\nu}} \text{ for } i \in [N].$$

Then $x^{i,\nu+1}$ is updated as follows for $i \in [N]$.

$$x^{i,\nu+1} = \Pi_{X^i}\left[x^{i,\nu} - \gamma_{i,\nu}\left(\nabla_{x^i}\theta_i(x^\nu) + w_{i,\nu}\right)\right], \qquad \forall\, i \in [N]. \qquad \textbf{(SGR)}$$

III. Termination test. If $\nu > K$, terminate; else $\nu := \nu + 1$ and return to (**I**).

Gradient-Response Schemes under Strong Monotonicity of F II

Suppose $\mathcal{F}_\nu$ denotes the history at iteration ν, defined as

$$\mathcal{F}_\nu \triangleq \left\{ x^0, \left\{ \left\{ \nabla_{x^i} \tilde{\theta}_i(x^0, \xi^{0,j}) \right\}_{j=1}^{M^{i,0}} \right\}_{i=1}^{N}, \cdots, \left\{ \left\{ \nabla_{x^i} \tilde{\theta}_i(x^{\nu-1}, \xi^{\nu-1,j}) \right\}_{j=1}^{M^{i,\nu-1}} \right\}_{i=1}^{N} \right\}.$$

Assumption 2

The steplength sequences $\{\gamma_{i,\nu}\}_{\nu=0}^{\infty}$ *satisfy one of the following for* $i \in [N]$.

(a) $\gamma_{i,\nu} = \gamma_i$ *for every* $\nu \geq 0$.

(b) $\sum_{\nu=0}^{\infty} \gamma_{i,\nu} = \infty$ *and* $\sum_{\nu=0}^{\infty} \gamma_{i,\nu}^2 < \infty$.

(c) For any ν*, the following holds.* $\gamma_{1,\nu} \leq \cdots \leq \gamma_{N,\nu} \leq \left(1 + \frac{\eta}{2L}\right) \gamma_{1,\nu}$. □

Gradient-Response Schemes under Strong Monotonicity of F III

Proposition 67 (Almost-sure convergence)

Suppose $M^{\nu,i} \geq M^{\nu} \geq 1$ for all ν and $i \in [N]$ and the steplength sequences satisfy Ass. 2 (b), (c). Then the following relation holds almost surely for all $\nu \geq 0$:

$$\mathbb{E}\left[\|x_{\nu+1} - x^*\|^2 \mid \mathcal{F}_\nu\right] \leq \left(1 - 2\gamma_{1,\nu}\eta + 2\gamma_{N,\nu}^2 L^2 + 2(\gamma_{N,\nu} - \gamma_{1,\nu})L\right) \|x_\nu - x^*\|^2 + \frac{\gamma_\nu^2 c^2}{M^\nu}. \quad (71)$$

Furthermore, $\{x_\nu\}$ converges almost surely to x^*, the unique NE. □

Gradient-Response Schemes under Strong Monotonicity of F IV

Theorem 68 (Rate Statements)

(i) Suppose the steplength sequences satisfy Ass. 2 (b) and (c). Suppose for $\nu \geq 0$ and for $i \in [N]$, $M^{i,\nu} = 1$ and $\gamma_{1,0} = 1/2\eta$. Then for $\nu \geq 2$,

$$\mathbb{E}\left[\|x_\nu - x^*\|^2\right] \leq \mathcal{O}\left(\frac{1}{\nu}\right). \qquad (\text{Optimal rate}) \tag{72}$$

(ii) Suppose the steplength sequences satisfy Ass. 2 (a) and (c). Suppose for $\nu \geq 0$ and for $i \in [N]$, $M^\nu = M_i^0 \lceil \rho^{-(\nu+1)} \rceil$, $\gamma_{i,0} = \gamma_i$, where

$$\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_N \leq \beta\gamma_1,$$

$\beta = \left(1 + \frac{\eta}{2L}\right)$, $\tilde{L}^2 = 2\beta^2 L^2$, and $\gamma_1 \leq \min\left\{1/\eta, 2\eta/\tilde{L}^2\right\}$. Then for $\tilde{q} \in (\rho, 1)$,

$$\mathbb{E}\left[\|x^\nu - x^*\|^2\right] \leq \begin{cases} C(\rho, q)\left(\max\{\rho, q\}\right)^\nu, & q \neq \rho \\ \tilde{D}\tilde{q}^\nu. & q = \rho \end{cases} \qquad (\text{Optimal det. rate}) \tag{73}$$

Gradient-Response Schemes under Strong Monotonicity of F V

- **Comments.**
  - When $\gamma_{i,\nu}$ are diminishing sequences and sample-size $M_{i,\nu} = 1$, the rate and complexity statements are optimal; matching statements for stochastic approximation for strongly convex stochastic optimization.
  - When $\gamma_{i,\nu}$ are constant and sample-size $M_{i,\nu}$ increases at a geometric rate, the rate statements match optimal deterministic (geometric) rates while the sample-complexity continues to be optimal.

$$\underbrace{[\,\text{Iteration complexity} \;= \mathcal{O}(1/\epsilon)\,]}_{M_{i,\nu}=1} \qquad \underbrace{[\,\text{Iteration complexity} \;= \mathcal{O}(\ln(1/\epsilon))\,]}_{M_{i,\nu} \text{ grows geometrically}}$$

Gap functions I

- Recall that x is an NE if and only if x is a solution of VI(X, F).
- However, when F is merely monotone, SOL(X, F) is not necessarily a singleton
- In this context, gap functions have been widely used [Larsson, 1994].
- Suppose G is a function defined as

$$G(x) \triangleq \sup_{y \in \mathcal{X}} (x - y)^\top F(y), \quad \forall x \in \mathcal{X}.$$

- G has two properties: $G(x)$ is nonnegative for any $x \in \mathcal{X}$ and $G(x) = 0$ if and only if x solves VI$(\mathcal{X}, F)$.

Regularized Gradient Response Schemes under monotonicity of F I

Regularized Gradient-Response Scheme

I. Initialization. For $i \in [N]$, choose $x^{i,0} \in X^i$, positive steplength sequence $\{\gamma_{i,\nu}\}$, regularization sequence $\{\eta_{i,\nu}\}$, and batch-size sequence $\{M^{i,\nu}\}$. Let $\nu := 0$.

II. General iteration. Suppose $w_\nu = (w_{i,\nu})_{i=1}^N$, where

$$w_{i,\nu} = \sum_{j=1}^{M^{i,\nu}} \frac{\left(\nabla_{x^i}\tilde{\theta}_i(x^\nu, \xi^{\nu,j}) - \nabla_{x^i}\theta_i(x^\nu)\right)}{M^{i,\nu}} \text{ for } i \in [N].$$

Then $x^{i,\nu+1}$ is updated as follows for $i \in [N]$.

$$x^{i,\nu+1} = \Pi_{X^i}\left[x^{i,\nu} - \gamma_{i,\nu}\left(\nabla_{x^i}\theta_i(x^\nu) + w_{i,\nu} + \eta_{i,\nu}x^{i,\nu}\right)\right], \forall i \in [N]. \quad \textbf{(R-GR)}$$

III. Termination test. If $\nu > K$, terminate; else $\nu := \nu + 1$ and return to (**I**).

Regularized Gradient Response Schemes under monotonicity of F II

Assumption 3

For each $i \in \mathcal{N}$, let the sequences $\{\gamma_{i,\nu}\}$ and $\{\eta_{i,\nu}\}$ be deterministic and monotonically decreasing to zero. Furthermore, the following hold.

(a) $\lim_{\nu \to \infty} \frac{\gamma^2_{\max,\nu}}{\gamma_{\min,\nu}\eta_{\min,\nu}} = 0$ and $\lim_{\nu \to \infty} \frac{\gamma_{\max,\nu} - \gamma_{\min,\nu}}{\gamma_{\min,\nu}\eta_{\min,\nu}} = 0$;

(b) $\sum_{\nu=0}^{\infty} \gamma_{\min,\nu}\eta_{\min,\nu} = \infty$; (c) For each $i \in \mathcal{N}$, $\sum_{\nu=0}^{\infty} \gamma^2_{i,\nu} < \infty$.

(d) $\sum_{\nu=0}^{\infty} \left(1 + \frac{1}{\gamma_{\min,\nu}\eta_{\min,\nu}}\right) \frac{\left(\eta_{\max,\nu-1} - \eta_{\min,\nu}\right)^2}{\eta^2_{\min,\nu}} < \infty$.

Remark 1

$\gamma_{i,\nu} = (\nu + \lambda_i)^{-a}$, $\eta_{i,\nu} = (\nu + \delta_i)^{-b}$ with $(1/2, 1) \ni a > b > 0$ and $a + b < 1$.

Regularized Gradient Response Schemes under monotonicity of F III

Theorem 69 (Almost Sure Convergence)

Let the iterate $\{x^{i,\nu}\}$ be generated by (R-GR) for $i \in \mathcal{N}$. Then the following statement holds almost surely.
If $\limsup_{\nu \to \infty} \frac{\eta_{\max,\nu}}{\eta_{\min,\nu}} = 1$, *then* $\{x_k\}$ *converges to the least-norm Nash equilibrium, i.e,* $\lim_{\nu \to \infty} x^\nu = x^*$ *with* $x^* = \mathrm{argmin}_{x \in \mathcal{X}^*} \|x\|$.

Theorem 70 (Rate and Complexity)

For each $i \in \mathcal{N}$, *we set* $\alpha_{i,k} = (k + \lambda_i)^{-a}$ *and* $\eta_{i,k} = (k + \delta_i)^{-b}$ *for some* $\lambda_i > 0, \delta_i > 0$ *with* $a \in (1/2, 1)$, $a > b$ *and* $a + b < 1$. *Then*

$$\mathbb{E}[G(\hat{x}_K)] = \mathcal{O}\left(K^{-1/2+\delta}\right), \quad \forall K \geq 1. \tag{74}$$

In addition, let $\widehat{x}_K$ *represent an* ϵ*-equilibrium, i.e.* $\mathbb{E}\left[\, G(\widehat{x}_K) \,\right] \leq \epsilon$. *Then the sample complexity (# of sampled gradients) or iteration complexity (# of projected grad steps) required to compute an* ϵ*-equilibrium is* $\mathcal{O}\left((N/\epsilon)^{2+\delta} \right)$.

Problem Statement I

Aggregative Game

$$\min_{x_i \in X_i} f_i(x_i, \bar{x}) \triangleq \mathbb{E}\left[\psi_i(x_i, \bar{x}; \xi)\right]$$

- $\mathcal{N} = \{1, \cdots, n\}$ is a group of n players, indexed by i;
- X_i is the strategy set of player i, $x = (x_1, \cdots, x_N)$ is a *strategy profile*;
- player i has an objective $f_i(x_i, \bar{x})$, where $\bar{x} \triangleq \sum_{i=1}^n x_i$ is the aggregate;
- $\xi : \Omega \to \mathbb{R}^m$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

- Nash equilibrium (NE) is a strategy profile $x^* = \{x_i^*\}_{i=1}^n$ such that no player can improve by unilateral deviation.
- Convexity of subproblems: X_i is a closed, compact, convex set; For any $y \in \mathbb{R}^d$, $f_i(x_i, y)$ is C^1 and convex in $x_i \in X_i$.
- Existence of a stochastic oracle returning a sampled gradient $\nabla_{x_i}\psi_i(x_i, y; \xi)$, $\nabla_{x_i} f_i(x_i, y) = \mathbb{E}[\nabla_{x_i} \psi_i(x_i, y; \xi)]$ and $\mathbb{E}[\|\nabla_{x_i} f_i(x_i, y) - \nabla_{x_i} \psi_i(x_i, y; \xi)\|^2] \leq \nu_i^2$.
- Existence of a stochastic oracle that returns a sampled gradient $\nabla_{x_i}\psi_i(x_i, y; \xi)$

Previous Work

- Consensus and Distributed Optimization.
  [Olfati-Saber and Murray, 2004] [Ren, Beard and Atkins, 2005] [Nedić and Ozdaglar, 2009], [Nedić, Ozdaglar and Parrilo, 2010], [Nedić and Olshevsky, 2015]
- Distributed schemes for Nash games
  - Gradient response+consensus for aggregative games [Koshal et al., 2016]
  - Aggregative games with coupling constraints [Paccagnan et al., 2017] [Belgioioso et al., 2017], a semi-dencentralized algorithm, requiring a *central node* for the update of the common multiplier.
  - Generalized Nash equilibrium problems:
    - Distributed primal-dual algorithms [Zhu and Frazzoli, 2017; Yi and Pavel, 2017].
    - Distributed stochastic gradient scheme with constant stepsize [Yu et al., 2017], mean-squared convergence to a neighborhood of the GNE.

Our Work

- The players cannot observe all rival strategies, while they can interact through a communication graph (connected) $\mathcal{G} = (\mathcal{N}, \mathcal{E}, A)$:

  - $\mathcal{E}$ is a collection of undirected edges;
  - Neighbor set $\mathcal{N}_i = \{j : (i,j) \in \mathcal{E}\}$;
  - The adjacency matrix $A = [a_{ij}]_{i,j=1}^n$, where $a_{ij} > 0$ if $j \in \mathcal{N}_i$ and $a_{ij} = 0$ otherwise such that A is doubly stochasticity.
- We aim to design a fully distributed algorithm to compute an NE only through local communications and computation.
- Can we achieve the best known deterministic rates?

Distributed PGR under strong monotonicity of F

$x_{i,k}$: its equilibrium strategy, $v_{i,k}$: the average of the aggregate.

Distributed Variance-reduced PGR

Initialize: Set $k = 0$, and $v_{i,0} = x_{i,0} \in X_i$ for any $i \in \mathcal{N}$.
Iterate until convergence

Consensus (average among neighbors). $\hat{v}_{i,k} := v_{i,k}$ and repeat τ_k times

$$\hat{v}_{i,k} := \sum_{j \in \mathcal{N}_i} a_{ij} \hat{v}_{j,k} \quad \forall i \in \mathcal{N} \qquad \text{or compact form } \widehat{V}_k = A^{\tau_k} V_k.$$

Strategy Update (walk along the negative gradient of the payoff).

$$x_{i,k+1} := \Pi_{X_i}\left[x_{i,k} - \frac{\alpha}{S_k} \sum_{p=1}^{S_k} \nabla_{x_i} \psi_i \left(x_{i,k}, n\hat{v}_{i,k}; \xi_k^p\right)\right] \quad \text{(Strategy update)}$$

$$v_{i,k+1} := v_{i,k} + x_{i,k+1} - x_{i,k}. \quad \text{(Update belief of aggregate)}$$

Analysis Sketch

F is η_ϕ-strongly monotone and L_ϕ-Lipschitz.

- Consensus error: based on $\left|\left[A^k\right]_{ij} - \frac{1}{n}\right| \leq \theta\beta^k$ for a constant $\theta > 0$ and $\beta \in (0,1)$, by defining $y_k \triangleq \sum_{i=1}^n v_{i,k}/n$ and $D_X \triangleq \sum_{j=1}^n \max_{x_j \in X_j} \|x_j\|$,

$$\|y_k - \hat{v}_{i,k}\| \leq \theta D_X \beta^{\sum_{p=0}^k \tau_p} + 2\theta D_X \sum_{s=1}^k \beta^{\sum_{p=s}^k \tau_p} \quad \forall k \geq 0.$$

- Suppose $\phi(x) \triangleq \left(\nabla_{x_i} f_i(x_i, \sum_{i=1}^n x_i)\right)_{i=1}^n$ is η_ϕ-strongly monotone and L_ϕ-Lipschitz continuous. Recursion on the conditional MSE.

$$\mathbb{E}[\|x_{k+1} - x^*\|^2 | \mathcal{F}_k] \leq \underbrace{\left(1 - 2\alpha\eta_\phi + 2\alpha^2 L_\phi^2\right)}_{\text{contraction property}} \|x_k - x^*\|^2 + \alpha^2 \underbrace{\sum_{i=1}^n \nu_i^2 / S_k}_{\text{noise}}$$

$$\underbrace{+ 4\alpha n D_X \sum_{i=1}^n L_i \|\hat{v}_{i,k} - y_k\| + 2\alpha^2 n^2 \sum_{i=1}^n L_i^2 \|\hat{v}_{i,k} - y_k\|^2}_{\text{consensus error}}.$$

Convergence Results–Geometric

Theorem 71 (Linear rate of convergence)

Set $\tau_k = k + 1$, $S_k = \lceil \rho^{-(k+1)} \rceil$ *for some* $\rho \in (0,1)$. *Suppose* $\alpha \in (0, \eta_\phi / L_\phi^2)$, *define* $\varrho_\phi \triangleq 1 - 2\alpha\eta_\phi + 2\alpha^2 L_\phi^2$ *and* $\gamma \triangleq \max\{\rho, \beta\}$. *Then*

$$\mathbb{E}[\|x_k - x^*\|^2] = \mathcal{O}(\max\{\varrho_\phi, \gamma\}^k).$$

Theorem 72 (Complexity Bounds)

Set $\tau_k = k + 1$, $\alpha = \frac{\eta_\phi}{2L_\phi^2}$ *and* $S_k = \lceil \rho^{-(k+1)} \rceil$ *with* $\rho \triangleq \max\left\{1 - \frac{\eta_\phi^2}{2L_\phi^2}, \beta\right\}$. *For obtaining* $\epsilon-$*NE such that* $\mathbb{E}[\|x_K - x^*\|^2] \leq \epsilon$, *the iteration complexity* $K = \mathcal{O}(\ln(1/\epsilon))$ *(optimal, deterministic), communication complexity* $\sum_{k=0}^K \tau_k = \mathcal{O}(\ln^2(1/\epsilon))$, *and the oracle complexity is* $\sum_{k=0}^K S_k = \mathcal{O}(1/\epsilon)$ *(optimal, SGD).*

less projections and communications than SGD $\mathcal{O}(1/\epsilon)$

best known comm. comp. in dis. opt. is $K \ln(K)$[Jakovetic, Xavier, and Moura, 14]

Convergence Results–Polynomial

- We might not increase the samples too fast when the oracle is costly.
- Explore the performance with slower increasing sample-size?

Theorem 73 (Polynomial rate of convergence)

Set $\tau_k = \lceil (k+1)^u \rceil$ *and* $S_k = \lceil (k+1)^v \rceil$ *for some* $u \in (0,1)$ *and* $v > 0$. *Let* $\alpha \in \left(0, \eta_\phi / L_\phi^2\right)$ *and define* $\varrho_\phi \triangleq 1 - 2\alpha\eta_\phi + 2\alpha^2 L_\phi^2$. *Then we obtain a polynomial rate of convergence* $\mathbb{E}[\|x_{k+1} - x^*\|^2] = \mathcal{O}\left((k+1)^{-v}\right)$,

Theorem 74 (Complexity Bounds)

Set $\tau_k = \lceil (k+1)^u \rceil$ *and* $S_k = \lceil (k+1)^v \rceil$ *for some* $u \in (0,1)$ *and* $v > 0$. *Then the iteration, communication, and oracle complexity to obtain an* $\epsilon-$*NE are bounded by* $\mathcal{O}((1/\epsilon)^{1/v})$, $\mathcal{O}((1/\epsilon)^{(u+1)/v})$, *and* $\mathcal{O}\left((1/\epsilon)^{1+1/v}\right)$, *respectively.*

Regularized Distributed GR for monotone F I

Distributed Nash Equilibrium Computation via Iterative Tik. Reg.

Initialize: Let $x_{i,0} \in \mathcal{X}_i$ and $v_{i,0} = h_i(x_{i,0})$ for each $i \in \mathcal{N}$. Let ν be a positive integer. Let $\{\gamma_{i,\nu}\}$ and $\{\eta_{i,\nu}\}$ be the positive sequences of steplengths and regularization parameters used by player i.

Iterate until $\nu > K$

Consensus. Player $i \in \mathcal{N}$ receives the neighbors' estimates $v_{j,\nu}, j \in \mathcal{N}_{i,\nu}$ and computes an intermediate estimate by

$$\hat{v}_{i,\nu} = \sum_{j \in \mathcal{N}_{i,\nu}} w_{ij,\nu} v_{j,\nu}. \tag{75}$$

Strategy Update. Player $i \in \mathcal{N}$ updates strategy and agg. belief as follows.

$$x_{i,\nu+1} = \Pi_{X_i}\left[x_{i,\nu} - \gamma_{i,\nu}\left(\nabla_{x^i}\theta_i(x_{i,\nu}, N\hat{v}_{i,\nu}) + \zeta_{i,\nu} + \eta_{i,\nu} x_{i,\nu}\right)\right], \tag{76}$$

$$v_{i,\nu+1} = \hat{v}_{i,\nu} + h_i(x_{i,\nu+1}) - h_i(x_{i,\nu}). \tag{77}$$

Consequently, the hisory $\mathcal{F}_\nu \triangleq \{x_0, \zeta_{i,l}, i \in \mathcal{N}, l = 0, 1, \cdots, \nu - 1\}$. Then by Algorithm 112 it is seen that $x^{i,\nu}$, $\hat{v}_{i,k,\nu}, i \in \mathcal{N}$ are adapted to $\mathcal{F}_\nu$.

We impose the following conditions on the time-varying communication graph $\mathcal{G}_\nu = \{\mathcal{N}, \mathcal{E}_\nu\}$.

Assumption 4

(a) W_ν *is doubly stochastic for any* $\nu \geq 0$*;*
(b) There exists a constant $0 < \varsigma < 1$ *such that for all* $\nu \geq 0$*,*

$$\omega_{ij,\nu} \geq \varsigma, \quad \forall (i,j) \in \mathcal{N} \times \mathcal{N}_{i,\nu}.$$

(c) There exists a positive integer P such that the graph given by the union $\{\mathcal{N}, \bigcup_{l=1}^{P} \mathcal{E}_{\nu+l}\}$ *is strongly connected for any* $\nu \geq 0$*.*

For any $\nu \geq 0$, let Φ be defined as $\Phi(k, k+1) = \mathbf{I}_N$, where $\mathbf{I}_N \in \mathbb{R}^{N \times N}$ denotes the identity matrix. We introduce the transition matrix $\Phi(k, s) = W_k W_{k-1} \cdots W_s$ from $s \geq 0$ to $k \geq s$.

Lemma 75

Let Assumption 4 hold. Then there exist constants $\theta = \left(1 - \frac{\varsigma}{4N^2}\right)^{-2} > 0$ *and* $\beta = \left(1 - \frac{\varsigma}{4N^2}\right)^{1/P} \in (0,1)$ *such that for any* $k \geq s \geq 0$:

$$\left|[\Phi(k,s)]_{ij} - \frac{1}{N}\right| \leq \theta\beta^{k-s}, \quad \forall i,j \in \mathcal{N}. \tag{78}$$

Proposition 76 (Consensus bound)

Let the constants θ *and* β *be given in Lemma 75. Then for each player* $i \in \mathcal{N}$ *and all* $\nu \geq 0$,

$$\begin{aligned} \|\sigma(x^\nu) - N\hat{v}_{i,\nu}\| \leq \theta M_H N \beta^\nu & \\ + \theta N \sum_{s=1}^{\nu} \beta^{\nu-s}\alpha_{\max,s-1} \sum_{j=1}^{N} L_{hj}(C + \eta_{\max,s-1}M_j + \|\zeta_{j,s-1}\|). & \end{aligned} \tag{79}$$

Theorem 77 (Almost Sure Convergence of Dist. R-GR)

The following statements holds almost surely. If $\limsup_{\nu\to\infty} \frac{\eta_{\max,\nu}}{\eta_{\min,\nu}} = 1$, *then* $\{x^\nu\}$ *converges to the least-norm Nash equilibrium, i.e,*

$$\lim_{\nu\to\infty} x^\nu = x^* \quad \text{with} \quad x^* = \operatorname*{argmin}_{x\in\mathcal{X}^*} \|x\|.$$

Theorem 78 (**Convergence rate of Algorithm 112)**

For each $i \in \mathcal{N}$, *let* $\alpha_{i,\nu} = (\nu + \lambda_i)^{-a}$ *and* $\eta_{i,\nu} = (\nu + \delta_i)^{-b}$ *for some* $\lambda_i > 0, \delta_i > 0$ *with* $a \in (1/2, 1),\ a > b$ *and* $a + b < 1$. *Then*

$$\mathbb{E}[G(\hat{x}_K)] = \mathcal{O}\big(NK^{-b}\big) + \mathcal{O}\big(N^5\beta^{-1}(1-\beta)^{-2}K^{a-1}\big), \quad \forall K \geq 1. \tag{80}$$

Corollary 79 (**Rate and complexity of Dist. R-GR)**

For each $i \in \mathcal{N}$, we set $\gamma_{i,\nu} = (\nu + \lambda_i)^{-(1/2+\tau)}$ and $\eta_{i,k} = (\nu + \delta_i)^{-(1/2-2\tau)}$ for some $\lambda_i > 0, \delta_i > 0$ with $0 < \tau < 1/4$. Then for any $K \geq 1$,

$$\mathbb{E}[G(\hat{x}_K)] = \mathcal{O}\big(NK^{-(0.5-2\tau)}\big) + \mathcal{O}\big(N^5\beta^{-1}(1-\beta)^{-2}K^{-(0.5-\tau)}\big). \quad (81)$$

In addition, let $\widehat{x}_K$ represent an ϵ-equilibrium, i.e. $\mathbb{E}\left[\, G(\widehat{x}_K)\,\right] \leq \epsilon$. Then the sample complexity (number of sampled gradients) or equivalently iteration complexity (number of proximal evaluations) required to compute an ϵ-equilibrium is no smaller than

$$\max\left\{\mathcal{O}\left((N/\epsilon)^{\frac{1}{0.5-2\tau}}\right), \mathcal{O}\left((N^5/(\epsilon\beta(1-\beta)^2))^{\frac{1}{0.5-\tau}}\right)\right\}.$$

We can further obtain a simplified sample (or iteration) complexity $\mathcal{O}\left(\left(\frac{1}{\epsilon}\right)^{2+\tilde{\varepsilon}}\right)$ for some $\tilde{\varepsilon} > 0$ when the impact of N and the network connectivity parameter β are neglected.

Basic Assumptions

- Convexity of subproblems
  - X_i is a closed, compact, convex set.
  - $f_i(x_i, x_{-i})$ is convex and C^2 in x_i over an open set containing X_i for any given $x_{-i} \in \prod_{j \neq i} X_j$.

- Existence of a stochastic first-order oracle (SFO):

For any $i \in \mathcal{N}$ and x, ξ, (SFO) returns a sampled gradient $\nabla_{x_i} \psi_i(x_i, x_{-i}; \xi)$ s.t.

  - Unbiased: $\nabla_{x_i} f_i(x_i, x_{-i}) = \mathbb{E}[\nabla_{x_i} \psi_i(x_i, x_{-i}; \xi(\omega))]$;
  - Bounded second moments: There exists $M_i > 0$ such that

$$\mathbb{E}[\|\nabla_{x_i} \psi_i(x_i, x_{-i}; \xi(\omega))\|^2] \leq M_i^2.$$

◆ Proximal best-response map

$$\widehat{x}(y) \triangleq \underset{x \in X}{\operatorname{argmin}} \left[\sum_{i=1}^{N} \mathbb{E}[\psi_i(x_i, y_{-i}; \omega)] + \frac{\mu}{2} \|x - y\|^2 \right], \quad \mu > 0$$

The objective function is separable in x_i, player i's subproblem is

$$\widehat{x}_i(y) \triangleq \underset{x_i \in X_i}{\operatorname{argmin}} \left[\mathbb{E}[\psi_i(x_i, y_{-i}; \omega)] + \frac{\mu}{2} \|x_i - y_i\|^2 \right].$$

◆ Fixed Point: $x^* = \widehat{x}(x^*)$; From [Facchinei and Pang, 2009]

,

- x^* is an NE iff x^* is a fixed point of the proximal map $\widehat{x}(\bullet)$
- $x_{k+1} = \widehat{x}(x_k)$ converges linearly to x^* when $\widehat{x}(\bullet)$ is contractive

Sufficient conditions for contractive PBR map:[Facch. & Pang, 2009]

- Define the $N \times N$ real matrix $\Gamma = [\gamma_{ij}]_{i,j=1}^{N}$:

$$\Gamma \triangleq \begin{pmatrix} \frac{\mu}{\mu+\zeta_{1,\min}} & \frac{\zeta_{12,\max}}{\mu+\zeta_{1,\min}} & \cdots & \frac{\zeta_{1N,\max}}{\mu+\zeta_{1,\min}} \\ \frac{\zeta_{21,\max}}{\mu+\zeta_{2,\min}} & \frac{\mu}{\mu+\zeta_{2,\min}} & \cdots & \frac{\zeta_{2N,\max}}{\mu+\zeta_{2,\min}} \\ \vdots & & \ddots & \\ \frac{\zeta_{N1,\max}}{\mu+\zeta_{N,\min}} & \frac{\zeta_{N2,\max}}{\mu+\zeta_{N,\min}} & \cdots & \frac{\mu}{\mu+\zeta_{N,\min}} \end{pmatrix}$$

  with
  $\zeta_{i,\min} \triangleq \inf_{x\in X} \lambda_{\min}\left(\nabla^2_{x_i} f_i(x)\right)$, and $\zeta_{ij,\max} \triangleq \sup_{x\in X} \|\nabla^2_{x_i x_j} f_i(x)\| \quad \forall j \neq i$
  measuring the coupling of players' subproblem.
- If the spectral radius $\rho(\Gamma) < 1$, then there exist a scalar $a \in (0,1)$ and monotonic norm $|\,\|\bullet\|\,|$ such that

$$\left|\left\|\begin{pmatrix} \|\widehat{x}_1(y') - \widehat{x}_1(y)\| \\ \vdots \\ \|\widehat{x}_N(y') - \widehat{x}_N(y)\| \end{pmatrix}\right\|\right| \leq a \left|\left\|\begin{pmatrix} \|y'_1 - y_1\| \\ \vdots \\ \|y'_N - y_N\| \end{pmatrix}\right\|\right|.$$

Challenges

- Closed-form expression of proximal best-response map is unavailable in finite time since the objective is expectation-valued

- We consider several *inexact* proximal best-response schemes
  - best-response solutions are approximated via stochastic approximation (SA)
  - and the inexactness can be driven to zero by an increasing number of projected gradient steps.

Algorithm Design I

Algorithm Synchronous inexact proximal best-response scheme

Set $k = 0$, $x_{i,0} \in X_i$; Let $\{\alpha_{i,k}\}_{k \geq 1}$ be a given sequence.

(1) For $i = 1, \ldots, N$, let $x_{i,k+1} \in X_i$ be defined as follows:

$$x_{i,k+1} = \widehat{x}_i(x_k) + \varepsilon_{i,k+1}$$

with $\{\varepsilon_{i,k+1}\}$ satisfying $\mathbb{E}\left[\|\varepsilon_{i,k+1}\|^2 \middle| \mathcal{F}_k\right] \leq \alpha_{i,k}^2$ *a.s.*, where $\mathcal{F}_k = \sigma\{x_0, \cdots, x_k\}$.

(2) $k := k + 1$; If $k < K$, return to (1); else STOP.

$$\widehat{x}_i(x_k) \triangleq \operatorname*{argmin}_{x_i \in X_i} \left[\mathbb{E}[\psi_i(x_i, x_{-i,k}; \omega)] + \frac{\mu}{2}\|x_i - x_{i,k}\|^2\right].$$

- Stochastic approximation (SA) to obtain an inexact best-response.

Algorithm Design II

$$z_{i,t+1} := \Pi_{X_i}\left[z_{i,t} - \gamma_t\left(\nabla_{x_i}\psi_i(z_{i,t}, x_{-i,k}; \xi^t_{i,k}) + \mu(z_{i,t} - x_{i,k})\right)\right], \qquad (\text{SA}_{i,k})$$

where $z_{i,1} = x_{i,k}$, $\gamma_{i,t} = 1/\mu(t+1)$. Set $x_{i,k+1} = z_{i,j_{i,k}}$.

Lemma 80 (Error Bounds of SA [Nemirovski et al., 2009])

Define $\xi_{i,k} = (\xi_{i,k}^1, \cdots, \xi_{i,k}^{j_{i,k}})$, *and* $\mathcal{F}_k = \sigma\{x_0, \xi_{i,l}, i \in \mathcal{N}, 0 \leq l \leq k-1\}$. *Assume that for any* $i \in \mathcal{N}$, *the random variables* $\{\xi_{i,k}^t\}_{1 \leq t \leq j_{i,k}}$ *are iid and the random vector* $\xi_{i,k}$ *is independent of* $\mathcal{F}_k$. *Then for any* $t \geq 1$ *we have*

$$\mathbb{E}\left[\|z_{i,t} - \widehat{x}_i(x_k)\|^2 \middle| \mathcal{F}_k\right] \leq \frac{Q_i}{(t+1)}, \quad a.s.$$

where $Q_i \triangleq \frac{2M_i^2}{\mu^2} + 2D_{X_i}^2$, *and* $D_{X_i} = \sup\{d(x_i, x_i') : x_i, x_i' \in X_i\}$.

$$\mathbb{E}\left[\|\varepsilon_{i,k+1}\|^2 \middle| \mathcal{F}_k\right] = \mathbb{E}\left[\|x_{i,k+1} - \widehat{x}_i(y_k)\|^2 \middle| \mathcal{F}_k\right] \leq \frac{Q_i}{j_{i,k}} =: \alpha_{i,k}^2$$

Almost Sure Convergence

Let the sequence $\{x_k\}_{k\geq 0}$ be generated by the synchronous algorithm. Assume that $\|\Gamma\| < 1$, and $\alpha_{i,k} \geq 0$ with $\sum_{k=1}^{\infty} \alpha_{i,k} < \infty$ for any $i \in \mathcal{N}$. Then for any $i \in \mathcal{N}$,

$$\lim_{k\to\infty} x_{i,k} = x_i^* \quad a.s.$$

Convergence in Mean and of the Variance

Let the sequence $\{x_k\}_{k=1}^{\infty}$ be generated by the synchronous algorithm. Assume that $\|\Gamma\| < 1$, and that $0 \leq \alpha_{i,k} \to 0$ as $k \to \infty$ for any $i \in \mathcal{N}$. Then for any $i \in \mathcal{N}$,

(a) (**convergence in mean**) $\lim_{k\to\infty} \mathbb{E}[\|x_{i,k} - x_i^*\|] = 0.$

(b) (**convergence of the variance of** x_k) $\lim_{k\to\infty} \mathbb{V}\mathrm{ar}(x_k) = 0.$

Geometric Convergence

Consider the synchronous scheme where $\mathbb{E}[\|x_{i,0} - x_i^*\|] \leq C \ \forall i \in \mathcal{N}$. Assume that $a = \|\Gamma\| < 1$, and that $\alpha_{i,k} = \eta^k \ \forall i \in \mathcal{V}$ with $\eta \in (0, 1)$. Define

$$u_k = \mathbb{E}\left[\left\| \begin{pmatrix} \|x_{1,k} - x_1^*\| \\ \vdots \\ \|x_{N,k} - x_N^*\| \end{pmatrix} \right\|\right].$$

Then, the following holds for $k \geq 0$

(a) If $\eta = a$, $q > a$ and $D \triangleq 1/\ln((q/a)^e)$, then $u_k \leq (u_0 + \sqrt{N}k)a^k \leq \sqrt{N}(C + D)q^k$.

(b) If $\eta \in (a, 1)$, then $u_k \leq \left(\sqrt{N}C + \frac{\sqrt{N}\eta}{\eta - a}\right) q^k$ with $q = \eta$.

(c) If $0 < \eta < a$, then $u_k \leq \left(\sqrt{N}C + \frac{\sqrt{N}a}{a - \eta}\right) q^k$ with $q = a$.

Overall iteration complexity

Consider the synchronous scheme and let inexact solutions be computed via SA, where $\mathbb{E}[\|x_{i,0} - x_i^*\|^2] \leq C^2$. Assume that $a = \|\Gamma\| < 1$ and $\alpha_{i,k} = \eta^k \ \forall i \in \mathcal{V}$ with $\eta \in (0,1)$. Then the number of projected gradient steps[1] for i to achieve an $\epsilon-$NE is no greater than $\mathcal{O}\left(\left(\frac{\sqrt{N}}{\epsilon}\right)^2 + \left(\ln\left(\frac{\sqrt{N}}{\epsilon}\right)\right)\right)$.

The bound grows slowly in # of players N, a desirable feature when faced by a large collection of players.

Comparison with Stochastic GR: Competitive portfolio Investment C. OCinneide, B. Scherer, and X. Xu (2006)

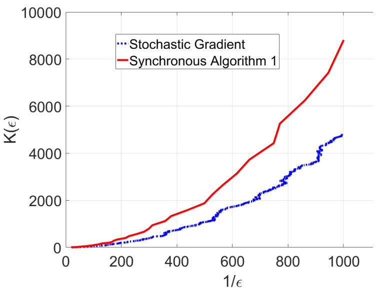


Figure: Empirical Iteration Complexity

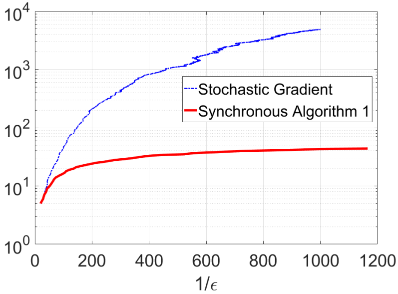


Figure: Empirical Commun. Complex.

- The iteration complexity is of the same order as stochastic gradient response (SGR); but the constant of SG is superior to that of the synchronous BR scheme.
- Significant decrease in communication overhead compared to SGR; communication overhead crucial in rendering a scheme impractical.

Asynchronous Scheme: Algorithm Design Bertsekas and Tsitsiklis (1989)

- **Motivation:**
  In a large-scale network, players might not be able to make simultaneous updates nor have access to their rivals' latest information.

- **Description:**
  - $T_i \subset T = \{0, 1, 2, \cdots\}$: the set of times player i updates x_i
  - $y_k^i \triangleq (x_{1,k-d_1^i(k)}, \cdots, x_{n,k-d_N^i(k)})$ is available to player i if $k \in T_i$, where $d_j^i(k)$ denotes the communication delay

- **Assumptions**
  - Almost Cyclic Rule: There exists an integer $B_1 > 0$ such that each player updates its decision at least once during any time interval of length B_1
  - Partial Asynchronism: There exists an integer $B_2 \geq 0$ such that

$$0 \leq d_j^i(k) \leq B_2 \quad \forall i, j = 1, \cdots, N, \; k \geq 0$$

Algorithm Asynchronous inexact proximal BR scheme

Let $k := 0$, $x_{i,0} \in X_i$ for $i = 1, \ldots, N$.

(1) For $i = 1, \ldots, N$, if $k \in T_i$, then set $y_k^i \triangleq (x_{1,k-d_1^i(k)}, \cdots, x_{n,k-d_N^i(k)})$.

(2) For $i = 1, \ldots, N$, if $k \in T_i$, then updates $x_{i,k+1} \in X_i$ as follows:

$$x_{i,k+1} = \widehat{x}_i(y_k) + \varepsilon_{i,k+1}$$

with $\varepsilon_{i,k+1}$ satisfying $\mathbb{E}\left[\|\varepsilon_{i,k+1}\|^2 \middle| \mathcal{F}_k\right] \leq \alpha_{i,k}^2$ *a.s.*, where $\mathcal{F}_k = \sigma\{x_0, \cdots, x_k\}$.
Otherwise, if $k \notin T_i$, then $x_{i,k+1} := x_{i,k}$.

(3) $k := k + 1$; If $k < K$, return to (1); else STOP.

Define $n_0 = \lceil \frac{B_2}{B_1} \rceil$, let $\beta_{i,k}$ denote the number of elements in T_i that are not larger than k.

Lemma 7.1 (Linear Rate of Convergence)

Let the asynchronous inexact proximal best-response scheme be applied to the N-player stochastic Nash game, where $\alpha_{i,k+1} = \eta^{\beta_{i,k}}$ for some $\eta \in (0,1)$, and $\mathbb{E}[\|x_{i,0} - x_i^*\|] \leq C \ \forall i \in \mathcal{N}$. Assume $a = \|\Gamma\|_\infty < 1$. If $q > c \triangleq \rho^{\frac{1}{B_1}}$ and $D > 1/\ln((q/c)^e)$,

$$\max_{i \in \mathcal{N}} \mathbb{E}[\|\widehat{x}_{i,k} - x_i^*\|] \leq \rho^{-\frac{B_1 - 1}{B_1}} (C + D) q^k \quad \forall k \geq 0,$$

Iteration Complexity (Impact of delay and asynchronicity)

Consider the asynchronous algorithm and let the inexact proximal solutions be computed via SA, where $\alpha_{i,k+1} = \eta^{\beta_{i,k}}$ for $\eta \in (0,1)$. Suppose $a = \|\Gamma\|_\infty < 1$. Then the number of projected gradient steps[2] for i to compute an $\epsilon-$NE is no greater than $\mathcal{O}\left((1/\epsilon)^{2B_1(1+\lceil \frac{B_2}{B_1} \rceil)+\delta}\right)$.

update	delay	complexity bound
B_1	B_2	$\mathcal{O}\left((1/\epsilon)^{2B_1(1+\lceil \frac{B_2}{B_1} \rceil)+\delta}\right)$
1	B_2	$\mathcal{O}\left((1/\epsilon)^{2(1+B_2)+\delta}\right)$
1	0	$\mathcal{O}\left((1/\epsilon)^{2+\delta}\right)$

Set $B_1 = 1$, the communication delays $k - \tau_j^i(k)$ are independently generated from a uniform distribution on the set $\{0, 1, \cdots, B_2\}$.

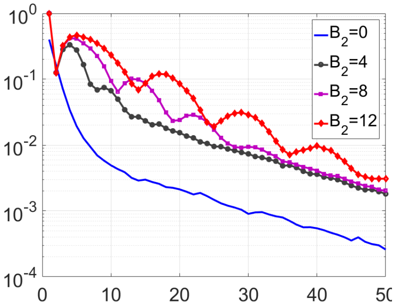


Figure: Linear Convergence

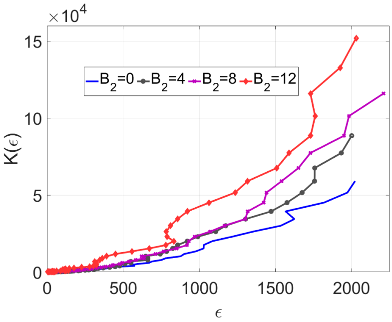


Figure: Empirical Iteration Complexity

Randomized Best-Response Scheme I

Literature review

- The randomized block-coordinate descent method **[Y. Nestrerov (2012)]** partitions the coordinates into several blocks and randomly choses a single block to update while the other blocks keep invariant at each iteration.
- Generalized to the fixed point problem by **[P. L. Combettes and J. C Pesquet (2015)]**, in which a subset of block variables is randomly updated

Randomized Best-response: For any $i \in \mathcal{N}$, let $\chi_{i,k} = 1$ (or 0) if player i updates at iteration k (or not).

◆ Assumption: For any $i \in \mathcal{N}$, $\mathbb{P}(\chi_{i,k} = 1) = p_i > 0$ and $\chi_{i,k}$ is independent of $\mathcal{F}_k$.

Randomized Best-Response Scheme II

Algorithm Randomized inexact proximal best-response scheme

Let $k := 0$, $x_{i,0} \in X_i$ for $i = 1, \ldots, N$.

(1) If $\chi_{i,k} = 1$, then $x_{i,k+1} \in X_i$ is defined as follows:

$$x_{i,k+1} = \widehat{x}_i(x_k) + \varepsilon_{i,k+1}$$

with $\varepsilon_{i,k+1}$ satisfying $\mathbb{E}\left[\|\varepsilon_{i,k+1}\|^2 \middle| \mathcal{F}_k\right] \leq \alpha_{i,k}^2$ *a.s.*, where $\mathcal{F}_k = \sigma\{x_0, \cdots, x_k\}$.
Otherwise, $x_{i,k+1} = x_{i,k}$ when $\chi_{i,k} = 0$.

(2) $k := k + 1$; If $k < K$, return to (1); else STOP.

Almost Sure Convergence

Let the sequence $\{x_k\}_{k\geq 0}$ be generated by the randomized algorithm. Assume that $a = \|\Gamma\| < 1$ and for any $i \in \mathcal{N}$, $0 \leq \alpha_{i,k} < 1$ and $\sum_{k=0}^{\infty} \alpha_{i,k} < \infty$ *a.s.* Then for any $i \in \mathcal{N}$, $\lim_{k\to\infty} x_{i,k} = x_i^*$ *a.s.*

Geometric Convergence

Let the sequence $\{x_k\}_{k\geq 0}$ be generated by the randomized algorithm.[3] Then the following holds for $k \geq 0$,

$$\mathbb{E}\left[\|x_k - x^*\|\right] \leq \sqrt{N}(\widetilde{C} + \tilde{D})\tilde{q}^k.$$

Overall Iteration Complexity

Let the randomized algorithm be applied with inexact solutions computed via SA, where $\alpha_{i,k} = \eta^{\beta_{i,k}+1}$ for some $\eta \in (0,1)$. Suppose $a = \|\Gamma\| < 1$. Then expected number of projected gradient steps[4] for i to compute an $\epsilon-$NE is no greater than $\mathcal{O}\left(\frac{\sqrt{N}p_{\max}}{\tilde{\epsilon}}\right)^{\frac{\ln(1/\tilde{\eta}_0^2)}{\ln(1/\tilde{q})}} + \left\lceil \frac{\ln(1/\tilde{\epsilon})}{\ln(1/\tilde{q})} \right\rceil$.

Summary of findings

Update scheme	Asymptotic convergence	Rate of convergence	Iteration complexity
Synchronous Algorithm (using $\|.\|_2$ norm)	a.s. convergence convergence in mean	geometric	ϵ-NE$_2$: $\mathcal{O}\left((\sqrt{N}/\epsilon)^{2+\delta}\right)$ $\eta \in (a,1)$: $\mathcal{O}(N/\epsilon^2)$
Randomized Algorithm (using $\|.\|_2$ norm)	a.s. convergence convergence in mean	geometric	ϵ-NE$_2$: $\mathcal{O}\left((\sqrt{N}/\epsilon)^{2\ln(\tilde{\eta}_0^{-1})/\ln(\tilde{\eta}^{-1})+\delta}\right)$
Asynchronous Algorithm (using $\|.\|_\infty$ norm)	convergence in mean	geometric	ϵ-NE$_\infty$: $\mathcal{O}\left((1/\epsilon)^{2B_1\left(1+\lceil\frac{B_2}{B_1}\rceil\right)+\delta}\right)$ $\mathcal{O}\left((1/\epsilon)^{2\left(1+\lceil\frac{B_2}{N}\rceil\right)+\delta}\right)$

Table: Summary of Contributions

- **Key findings: the iteration complexity is $\mathcal{O}(1/\epsilon^{2(1+c)+\delta})$**
  - $c = 0$ for the synchronous scheme
  - $c > 0$ represents the positive cost of randomization in the randomized scheme
  - $c > 0$ represents the positive cost of asynchronicity and delay

Our Work

- The players cannot observe rival strategies, while interacting through a communication graph (connected) $\mathcal{G} = (\mathcal{N}, \mathcal{E}, A)$:

- $\mathcal{E}$ is a collection of undirected edges;
- Neighbor set $\mathcal{N}_i = \{j : (i,j) \in \mathcal{E}\}$;
- The adjacency matrix $A = [a_{ij}]_{i,j=1}^n$, where $a_{ij} > 0$ if $j \in \mathcal{N}_i$ and $a_{ij} = 0$ otherwise such that A is doubly stochastic.

- We aim to design a fully distributed algorithm to compute an NE only through local communications and computation.
- Can we achieve the best known deterministic rates?

Distributed PBR

Distributed Variable Sample-size Proximal Best-response Scheme

Initialize: Set $k = 0$, and $v_{i,0} = x_{i,0} \in X_i$ for any $i \in \mathcal{N}$.

Iterate until convergence

Consensus. $\hat{v}_{i,k} := v_{i,k}\ \forall i \in \mathcal{N}$ and repeat τ_k times

$$\hat{v}_{i,k} := \sum_{j \in \mathcal{N}_i} a_{ij} \hat{v}_{j,k} \quad \forall i \in \mathcal{N}.$$

Strategy Update (sample average objective), for any $i \in \mathcal{N}$

$$x_{i,k+1} = \operatorname*{argmin}_{x_i \in X_i} \left[\frac{1}{S_k} \sum_{p=1}^{S_k} \psi_i(x_i, n\hat{v}_{i,k}; \xi_k^p) + \frac{\mu}{2} \|x_i - x_{i,k}\|^2 \right],$$

$$v_{i,k+1} := v_{i,k} + x_{i,k+1} - x_{i,k}.$$

Main Results

- Assumption: proximal BR map is contractive with parameter $a \in (0,1)$.

$$T_i(y) \triangleq \operatorname{argmin}_{x_i \in X_i} \left[f_i(x_i, \bar{y}) + \frac{\mu}{2}\|x_i - y_i\|^2 \right] \quad \mu > 0.$$

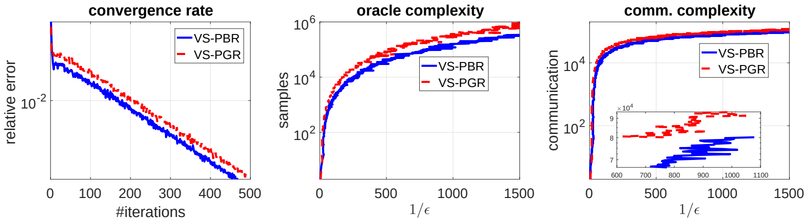
- Geometric Convergence. Set $\tau_k = k+1$ and $S_k = \lceil \eta^{-2k} \rceil$ with $\eta \in (0,1)$. Then $\mathbb{E}[\|x_k - x^*\|^2] = \mathcal{O}\left(\max\{a,\gamma\}^{2k}\right)$, where $\gamma \triangleq \max\{\eta, \beta\}$. The iteration, oracle, and communication complexity to compute an ϵ-NE are $\mathcal{O}(\ln(1/\epsilon))$, $\mathcal{O}(1/\epsilon)$, and $\mathcal{O}\left(\ln^2(1/\epsilon)\right)$, respectively.
- Often computing a sampled gradient is costly and geometric growth is impractical.

Polynomial growth in sample-size represents a “dial”.

- Polynomial Rate of Convergence. Set $\tau_k = \lceil (k+1)^u \rceil$ and $S_k = \lceil (k+1)^v \rceil$ for $u \in (0,1)$ and $v > 0$. Then $\mathbb{E}[\|x_{k+1} - x^*\|^2] = \mathcal{O}\left((k+1)^{-v}\right)$, the iteration, communication, and oracle complexity to obtain an $\epsilon-$NE are $\mathcal{O}((1/\epsilon)^{1/v})$, $\mathcal{O}((1/\epsilon)^{(u+1)/v})$, and $\mathcal{O}\left((1/\epsilon)^{1+1/v}\right)$, respectively.

Numerical Validation: Distributed VS-PGR and VS-PBR

Run both algorithms over a Erdős–Rényi graph with $\alpha = 0.04$, $\tau_k = k + 1$ and $S_k = \left\lceil 0.98^{-(k+1)} \right\rceil$, and $\mu = 30$.



Summary of Contributions

$$\min_{x_i \in \mathbb{R}^{d_i}} \; F_i(x_i, x_{-i}) \triangleq \mathbb{E}\left[\psi_i(x;\xi)\right] + r_i(x_i).$$

Algorithm	S_k	Rate $\mathbb{E}[\|x_k - x^*\|^2]$	Iter. Comp.	Oracle Comp.	Ass.
VS-PGR	$\lceil \rho^{-(k+1)} \rceil$	Linear: $\mathcal{O}(\rho^k)$	$\mathcal{O}(\ln(1/\epsilon))$	$\mathcal{O}(1/\epsilon)$	SM
	$\lceil (k+1)^v \rceil$	$\mathcal{O}(q^k) + \mathcal{O}(k^{-v})$	$\mathcal{O}((1/\epsilon)^{1/v})$	$\mathcal{O}(1/\epsilon)^{(1+1/v)}$	SM
VS-PBR	$\lceil \rho^{-(k+1)} \rceil$	$\mathcal{O}(\rho^k)$	$\mathcal{O}(\ln(1/\epsilon))$	$\mathcal{O}(1/\epsilon)$	CPM
	$\lceil (k+1)^v \rceil$	$\mathcal{O}(a^k) + \mathcal{O}(k^{-v})$	$\mathcal{O}(1/\epsilon^{1/v})$	$\mathcal{O}(1/\epsilon^{1+1/v})$	CPM

SM: Strongly monotone, CPM: Contract. prox. BR Map

Algorithm	S_k	Comm. τ_k	Rate $\mathbb{E}[\|x_k - x^*\|^2]$	Iter. Comp.	Oracle Comp.	Comm. Comp
d-VS-PGR	$\lceil \rho^{-(k+1)} \rceil$	$k+1$	Linear: $\mathcal{O}(\rho^k)$	$\mathcal{O}(\ln(1/\epsilon))$	$\mathcal{O}(1/\epsilon)$	$\mathcal{O}(\ln^2(1/\epsilon))$
	$\lceil (k+1)^v \rceil$	$\lceil (k+1)^u \rceil$	$\mathcal{O}((k+1)^{-v})$	$\mathcal{O}((1/\epsilon)^{1/v})$	$\mathcal{O}((1/\epsilon)^{1+1/v})$	$\mathcal{O}((1/\epsilon)^{\frac{1+u}{v}})$
d-VS-PBR	$\lceil \rho^{-(k+1)} \rceil$	$k+1$	Linear: $\mathcal{O}(\rho^k)$	$\mathcal{O}(\ln(1/\epsilon))$	$\mathcal{O}(1/\epsilon)$	$\mathcal{O}(\ln^2(1/\epsilon))$
	$\lceil (k+1)^v \rceil$	$\lceil (k+1)^u \rceil$	$\mathcal{O}((k+1)^{-v})$	$\mathcal{O}((1/\epsilon)^{1/v})$	$\mathcal{O}((1/\epsilon)^{1+1/v})$	$\mathcal{O}((1/\epsilon)^{\frac{1+u}{v}})$

(d-VS-PGR) and (d-VS-PBR) schemes for Aggregative games ($v > 0, u \in (0,1)$)

Summary and related work I

- **Existence of solutions to SVIs/SQVIs/SCPs.**
  - U. Ravat and UVS, *On the existence of solutions to stochastic quasi-variational inequality and complementarity problems.*, Math. Program. 165(1): 291-330 (2017)
  - U. Ravat and UVS, *On the characterization of solution sets of smooth and nonsmooth convex stochastic Nash games*, SIAM Journal of Optimization, Vol. 21, No. 3, Pg. 1168–1199, (2011).
  - U. Ravat and UVS, *On the characterization of solution sets of smooth and nonsmooth stochastic Nash games*, Proceedings of the American Control Conference (ACC), Baltimore, 2010.
- **Synch., asynch., and randomized BR schemes for stochastic Nash games**
  - UVS, Jong-Shi Pang, and Suvrajeet Sen, *Inexact best-response schemes for stochastic Nash games: Linear convergence and iteration complexity*, CDC 2016
  - Jinlong Lei, UVS, Jong-Shi Pang, and Suvrajeet Sen, *On Synchronous, Asynchronous, and Randomized Best-Response Schemes for Stochastic Nash Games*, Mathematics of Operations Research (2020).
- **Distributed schemes for Stochastic Nash games over graphs**
  - Jinlong Lei and UVS, *Linearly Convergent Variable Sample-Size Schemes for Stochastic Nash Games: Best-Response Schemes and Distributed Gradient-Response Schemes*, CDC 2018: 3547-3552
  - Jinlong Lei and UVS, *Distributed Variable Sample-Size Gradient-response and Best-response Schemes for Stochastic Nash Games over Graphs*, (SIOPT (2022))

Other research on computation of NE.

Summary and related work II

1. Computation of generalized Nash equilibria: Lemke-type schemes for computing non-VE GNEs [Schiro, Pang, and UVS, Math. Prog (2012)]; significant related work by Facchinei, Fischer, Kanzow, Tseng, etc., most recently by M. Ulbrich and S. Ulbrich.
2. Computation of equilibria of potential and weighted potential stochastic games via inexact synchronous and asynchronous BR schemes: [Lei and UVS, 2019]; has overlap with BCD schemes for nonconvex stochastic optimization

Usage of SA to obtain inexact solutions and derive overall complexity.

1. Stochastic ADMM schemes: [Xie and UVS, WSC 2016; TAC 2019]
2. Stochastic nonsmooth convex optimization: [Jalilzadeh, UVS, Blanchet, and Glynn], arxiv 2018
3. Stochastic generalized equations: [Cui and UVS, 2019]

Facchinei, F., Fischer, A., and Piccialli, V. (2007).
On generalized Nash games and variational inequalities.
Operations Research Letters, 35(2):159–164.

Facchinei, F. and Pang, J.-S. (2003).
Finite Dimensional Variational Inequalities and Complementarity Problems: Vols I and II.
Springer-Verlag, NY, Inc.

Harker, P. T. (1991).
Generalized Nash games and quasi-variational inequalities.
European Journal of Operational Research, 54(1):81–94.

Myerson, R. B. (1978).
Refinements of the Nash equilibrium concept.
International Journal of Game Theory, 7(2):73–80.

Nash, J. F. (1950).
Equilibrium points in n-person games.
Proceedings of National Academy of Science.

Nemirovski, A., Juditsky, A., Lan, G., and Shapiro, A. (2009).
Robust stochastic approximation approach to stochastic programming.

SIAM Journal on Optimization, 19(4):1574–1609.

Selten, R. (1975).
Reexamination of the perfectness concept for equilibrium points in extensive games.
International Journal of Game Theory, 4(1):25–55.

Shapiro, A., Dentcheva, D., and Ruszczyński, A. (2009).
Lectures on stochastic programming, volume 9 of *MPS/SIAM Series on Optimization*.
Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA.
Modeling and theory.