From Complementarity to Risk-averse stochastic equilibria: models and algorithms

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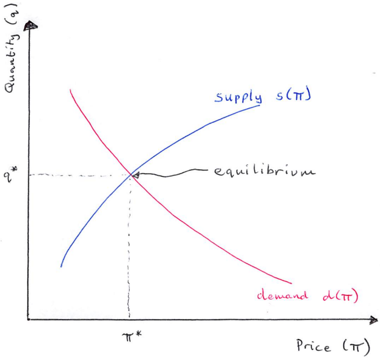
Equilibrium

- A state x of balance (often $F(x) = 0$)
- aequi + libra (equal weight): balance of competing influences

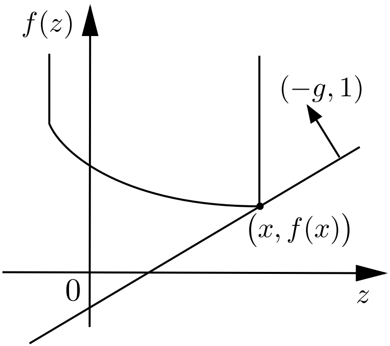
- Mechanics; Newton's Third Law: Action and Reaction
- Traffic; Wardrop's first principle: no driver unilaterally changes routes to improve his/her travel time

- Market/Economic: Economic forces such as supply and demand are balanced and in absence of external influences the (equilibrium) values will not change



Walras: $0 \leq s(\pi) - d(\pi) \perp \pi \geq 0$

Convex subdifferentials

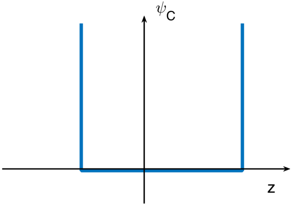


- Assume f is convex, then $f(z) \geq f(x) + \nabla f(x)^T (z - x)$ (linearization is below the function)
- Incorporate constraints by allowing f to take on $+\infty$ if constraint is violated $f : \mathbb{R}^n \mapsto (-\infty, +\infty]$
- $\partial f(x) = \{g : f(z) \geq f(x) + g^T (z - x), \forall z\}$, the subdifferential of f at x

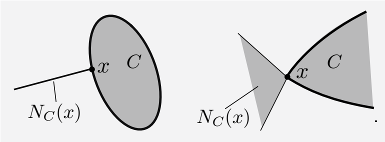
- If f is differentiable and convex, then $\partial f(x) = \{\nabla f(x)\}$
- e.g. $f(z) = \frac{1}{2} z^T Q z + p^T z$, then $\partial f(x) = \{Qx + p\}$
- x^* solves $\min f(x)$ if and only if $0 \in \partial f(x^*)$

Indicator functions and normal cones

$$\psi_{\mathcal{C}}(z) = \begin{cases} 0 & \text{if } z \in \mathcal{C} \\ \infty & \text{else} \end{cases}$$



$\psi_{\mathcal{C}}$ is a convex function when $\mathcal{C}$ is a convex set



If $x \in \mathcal{C}$, then

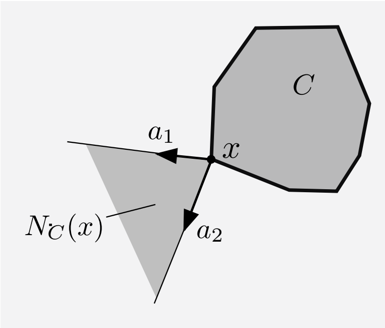
$$g \in \partial\psi_{\mathcal{C}}(x)$$
$$\iff \psi_{\mathcal{C}}(z) \geq \psi_{\mathcal{C}}(x) + g^T(z - x), \ \forall z$$
$$\iff 0 \geq g^T(z - x), \ \forall z \in \mathcal{C}$$

Normal cone to $\mathcal{C}$ at x,

$$N_{\mathcal{C}}(x) := \partial\psi_{\mathcal{C}}(x) = \begin{cases} \{g : g^T(z - x) \leq 0, \forall z \in \mathcal{C}\} & \text{if } x \in \mathcal{C} \\ \emptyset & \text{if } x \notin \mathcal{C} \end{cases}$$

Special cases and examples

- Normal cone is a cone
- $x \in \mathrm{int}(\mathcal{C})$, then $N_{\mathcal{C}}(x) = \{0\}$
- $\mathcal{C} = \mathbb{R}^n$, then $N_{\mathcal{C}}(x) = \{0\}, \forall x \in \mathcal{C}$



- $\mathcal{C} = \left\{z : a_i^T z \le b_i, i = 1, \ldots, m\right\}$ polyhedral
- $N_{\mathcal{C}}(x) = \left\{\sum_{i=1}^{m} \lambda_i a_i : 0 \le b_i - a_i^T x \perp \lambda_i \ge 0\right\}$
- $\perp$ makes product of items around it 0, i.e.

$$(b_i - a_i^T x)\lambda_i = 0, \;\; i = 1, \ldots, m$$

Some calculus

- $f_i : \mathbb{R}^n \mapsto (-\infty, \infty]$, $i = 1, \dots, m$, proper, convex functions

$$F = f_1 + \cdots + f_m$$

assume $\bigcap_{i=1}^{m} \operatorname{rint}(\operatorname{dom}(f_i)) \neq \emptyset$ then (as sets)

$$\partial F(x) = \partial f_1(x) + \cdots + \partial f_m(x), \ \forall x$$

- $\mathcal{C} = \bigcap_{i=1}^{m} \mathcal{C}_i$, then $\psi_{\mathcal{C}} = \psi_{\mathcal{C}_1} + \cdots + \psi_{\mathcal{C}_m}$, so $N_{\mathcal{C}} = N_{\mathcal{C}_1} + \cdots + N_{\mathcal{C}_m}$

$$x^* \text{ solves } \min_{x \in \mathcal{C}} f(x) \iff x^* \text{ solves } \min_x (f + \psi_{\mathcal{C}})(x)$$
$$\iff 0 \in \partial(f + \psi_{\mathcal{C}})(x^*) \iff 0 \in \nabla f(x^*) + N_{\mathcal{C}}(x^*)$$

Variational Inequality (replace $\nabla f(z)$ with $F(z)$)

- $F : \mathbb{R}^n \to \mathbb{R}^n$
- Ideally: $\mathcal{C} \subseteq \mathbb{R}^n$ – constraint set; Often: $\mathcal{C} \subseteq \mathbb{R}^n$ – simple bounds

$$\mathrm{VI}(F, \mathcal{C}) : \quad 0 \in F(z) + N_{\mathcal{C}}(z)$$

- VI generalizes many problem classes
- Nonlinear Equations: $F(z) = 0$ set $\mathcal{C} \equiv \mathbb{R}^n$
- Convex optimization: $F(z) = \nabla f(z)$
- For LP, set $F(z) \equiv \nabla f(z) = p$ and $\mathcal{C} = \{z : Az = a, Hz \leq h\}$.

Can use $\mathrm{VI}(F, \mathcal{C})$ to model an equilibrium

The positive orthant

Recall: If $\mathcal{C} = \left\{z : a_i^T z \leq b_i, i = 1, \ldots, m\right\}$ then

$$N_{\mathcal{C}}(x) = \left\{\sum_{i=1}^{m} \lambda_i a_i : 0 \leq b_i - a_i^T x \perp \lambda_i \geq 0\right\}$$

- $\mathcal{C} = \mathbb{R}^n_+ = \{z : -I_{i\cdot} z \leq 0, i = 1, \ldots, m\}$
- $0 \in F(z) + N_{\mathcal{C}}(z)$ iff $-F(z) = \sum_{i=1}^{m} -I_{i\cdot}^T \lambda_i = -\lambda$ where

$$0 \leq 0 + I_{i\cdot} x \perp \lambda_i \geq 0, \; i = 1, \ldots, n$$

- i.e.

$$0 \leq x_i \perp F_i(x) \geq 0, \; i = 1, \ldots, n$$

- More succinctly: $\mathrm{VI}(F, \mathbb{R}^n_+) \equiv \mathsf{NCP}(F)$:

$$0 \leq x \perp F(x) \geq 0$$

- For MCP (rectangular VI), set $\mathcal{C} \equiv [l, u]^n$.

Combining: KKT conditions (as MCP)

- Example: convex optimization first-order optimality condition:

$$
\begin{aligned}
x^* \text{ solves } \min_{x \in \mathcal{C}} f(x) &\iff 0 \in \nabla f(x^*) + N_{\mathcal{C}}(x^*) \\
&\iff 0 = \nabla f(x^*) + y,\ y \in N_{\mathcal{C}}(x^*) \\
&\iff 0 = \nabla f(x^*) + y,\ y = A^T \lambda, \\
&\qquad 0 \le b - Ax^* \perp \lambda \ge 0 \\
&\iff 0 = \nabla f(x^*) + A^T \lambda, \\
&\qquad 0 \le b - Ax^* \perp \lambda \ge 0
\end{aligned}
$$

- More generally, if $\mathcal{C} = \{z : g(z) \le 0\}$, g convex, (with CQ)

$$
\begin{aligned}
x^* \text{ solves } \min_{x \in \mathcal{C}} f(x) &\iff 0 \in \nabla f(x^*) + N_{\mathcal{C}}(x^*) \\
&\iff 0 = \nabla f(x^*) + \nabla g(x^*) \lambda, \\
&\qquad 0 \le -g(x^*) \perp \lambda \ge 0
\end{aligned}
$$

Other applications of complementarity

Complementarity can model fixed points and disjunctions

- Economics: Walrasian equilibrium (supply equals demand), taxes and tariffs, computable general equilibria, option pricing (electricity market), airline overbooking
- Transportation: Wardropian equilibrium (shortest paths), selfish routing, dynamic traffic assignment
- Applied mathematics: Free boundary problems
- Engineering: Optimal control (ELQP)
- Mechanics: Structure design, contact problems (with friction)
- Geology: Earthquake propagation

Good solvers exist for large-scale instances of Complementarity Problems

Equilibrium = the first-order optimality conditions (KKTs)

An equilibrium of a single optimization (a single agent) under CQs

$$\begin{array}{lll} \underset{x}{\text{minimize}} & f(x), & \\ \text{subject to} & g(x) \le 0, \ (\Rightarrow) & \\ & h(x) = 0, & \end{array} \qquad \begin{array}{r} \nabla f(x) - \nabla g(x)^T \lambda - \nabla h(x)^T \mu = 0, \\ 0 \ge g(x) \perp \lambda \le 0, \\ 0 = h(x) \perp \mu, \end{array}$$

- Mixed complementarity problem $\mathrm{MCP}(F, [l, u]) : l \le z \le u \ \perp \ F(z)$

Geometric first-order optimality conditions for a closed convex set C

$$\underset{x \in C}{\text{minimize}} \quad f(x), \ (\Rightarrow) \quad 0 \in \nabla f(x) + N_C(x)$$

$$\text{i.e. } \mathrm{VI}(\nabla f(x), C)$$

- Variational inequality $\mathrm{VI}(F, C) : \langle F(x), y - x \rangle \ge 0, \forall y \in C$

Generalizing to N agents: NEP

Nash equilibrium problem: $x = [x_i]_{i=1}^N$

$$\begin{array}{lll} \underset{x_i}{\text{minimize}} & f_i(x_i, x_{-i}), & \qquad \nabla_{x_i} f_i(x_i, x_{-i}) - \nabla g_i(x_i)\lambda_i - \nabla h_i(x_i)\mu_i = 0, \\ \text{subject to} & g_i(x_i) \le 0, \ (\Rightarrow) & \qquad 0 \ge g_i(x_i) \perp \lambda_i \le 0, \\ & h_i(x_i) = 0, & \qquad 0 = h_i(x_i) \perp \mu_i. \end{array}$$

- $x_{-i} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)^T$.
- Equilibrium: satisfy the KKT conditions of all agents simultaneously.
- Interactions occur only in objective functions.
- Example of an interaction: $f_i(x_i, x_{-i}) = c_i(x_i) - x_i p\left(\sum_{j=1}^N x_j\right)$

NEP + interacting feasible regions: GNEP

Generalized Nash equilibrium problem: $x = [x_i]_{i=1}^N$

$$\begin{array}{lll} \underset{x_i}{\text{minimize}} & f_i(x_i, x_{-i}), & \nabla_{x_i} f_i(x) - \nabla_{x_i} g_i(x)\lambda_i - \nabla_{x_i} h_i(x)\mu_i = 0, \\ \text{subject to} & g_i(x_i, x_{-i}) \le 0, \ (\Rightarrow) & 0 \ge g_i(x) \perp \lambda_i \le 0, \\ & h_i(x_i, x_{-i}) = 0, & 0 = h_i(x) \perp \mu_i. \end{array}$$

- Interactions occur in both objective functions and constraints.
- Interacting feasible regions:
  $K_i(x_{-i}) = \{x_i \in \mathbb{R}^{n_i} \mid g_i(x_i, x_{-i}) \le 0, h_i(x_i, x_{-i}) = 0\}$.
  - $K_i : \mathbb{R}^{n-n_i} \rightrightarrows \mathbb{R}^{n_i}$ a set-valued mapping
  - e.g., shared resources among agents: $\sum_{i=1}^N x_i \le b$, or strategic interactions
  - Quasi-variational inequality

(G)NEP + VI agent: MOPEC

Multiple optimization problems with equilibrium constraints:
$x = [x_i]_{i=1}^N, \pi$

$$\begin{array}{lll} \underset{x_i}{\text{minimize}} & f_i(x_i, x_{-i}, \pi), & \nabla_{x_i} f_i(x,\pi) - \nabla_{x_i} g_i(x,\pi)\lambda_i - \nabla_{x_i} h_i(x,\pi)\mu_i = 0, \\ \text{subject to} & g_i(x_i, x_{-i}, \pi) \leq 0, & 0 \geq g_i(x,\pi) \perp \lambda_i \leq 0, \\ & h_i(x_i, x_{-i}, \pi) = 0, & 0 = h_i(x,\pi) \perp \mu_i, \\ \hline \pi & \in \mathrm{SOL}(F, K), & \pi \in K(x), \langle F(\pi, x), y - \pi \rangle \geq 0, \ \forall y \in K(x). \end{array}$$

- No hierarchy between agents, c.f., MPECs and EPECs
- An example of a VI agent: market clearing conditions

$$0 \leq \text{supply} - \text{demand} \quad \perp \quad \text{price} \geq 0$$

Perfect competition (perfcomp)

$$\begin{array}{ll} \max_{x_i} \pi^T x_i - c_i(x_i) & \text{profit} \\ \text{s.t. } B_i x_i = b_i, x_i \geq 0 & \text{technical constr} \\ \hline 0 \leq \pi \perp \sum_i x_i - d(\pi) \geq 0 & \end{array}$$

- When there are many agents, assume none can affect π by themselves
- Each agent is a price taker
- Two agents, $d(\pi) = 24 - \pi$, $c_1 = 3$, $c_2 = 2$
- KKT(1) + KKT(2) + Market Clearing gives Complementarity Problem
- $x_1 = 0$, $x_2 = 22$, $\pi = 2$

Duopoly: two agents (cournot)

$$\max_{x_i} p(\sum_j x_j)^T x_i - c_i(x_i) \qquad \text{profit}$$

$$\text{s.t. } B_i x_i = b_i, x_i \geq 0 \qquad \text{technical constr}$$

- Cournot: assume each can affect π by choice of x_i
- Inverse demand $p(q)$: $\pi = p(q) \iff q = d(\pi)$
- Two agents, same data
- KKT(1) + KKT(2) gives Complementarity Problem
- $x_1 = 20/3$, $x_2 = 23/3$, $\pi = 29/3$
- Exercise of market power (some price takers, some Cournot)

How can you do this?

- Above models, and further examples are available for download at: `http://www.cs.wisc.edu/~ferris/empmodels`
- Implemented in GAMS (see more examples in EMP library)
- To run, can use GAMS Community License
- Users can request a free community license from `community@gams.com`. The community license lets you generate and solve linear models (LP, MIP, and RMIP) that do not exceed 5000 variables and 5000 constraints. For all other model types the model cannot be larger than 2500 variables and 2500 constraints.

- gams perfcomp
- gams cournot (or cournot1 for different syntax)
- gams harkredemp
- gams riskEx

General Equilibrium models (two3emp)

$$(C): \max_{x_k \in X_k} U_k(x_k) \text{ s.t. } p^T x_k \leq i_k(y, p)$$

$$(P): \max_{y_j \in Y_j} p^T g_j(y_j)$$

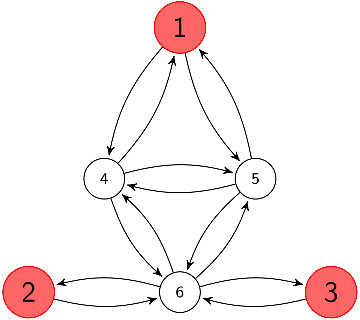
$$(M): \max_{p \geq 0} p^T \left(\sum_k x_k - \sum_k \omega_k - \sum_j g_j(y_j) \right) \text{ s.t. } \sum_l p_l = 1$$

$$\textit{or } 0 \leq \sum_k x_k - \sum_k \omega_k - \sum_j g_j(y_j) \perp p \geq 0$$

$$(I): i_k(y, p) = p^T \omega_k + \sum_j \alpha_{kj} p^T g_j(y_j)$$

This is an example of a MOPEC

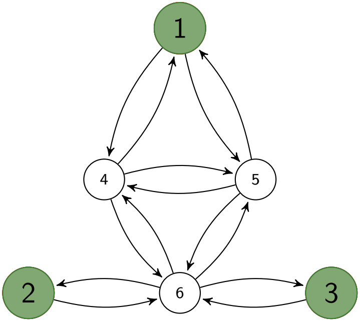
Spatial Price Equilibrium (harkredemp)



$n \in \{1, 2, 3, 4, 5, 6\}$
$L \in \{1, 2, 3\}$

Supply quantity: S_L
Production cost: $\Psi(S_L) = ..$

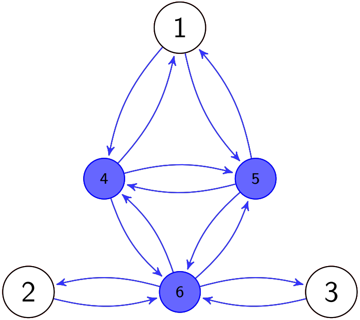
Spatial Price Equilibrium (harkredemp)



$n \in \{1, 2, 3, 4, 5, 6\}$
$L \in \{1, 2, 3\}$

Supply quantity: S_L
Production cost: $\Psi(S_L) = ..$
Demand: D_L
Unit demand price: $\theta(D_L) = ..$

Spatial Price Equilibrium (harkredemp)



$n \in \{1, 2, 3, 4, 5, 6\}$
$L \in \{1, 2, 3\}$

Supply quantity: S_L
Production cost: $\Psi(S_L) = ..$
Demand: D_L
Unit demand price: $\theta(D_L) = ..$
Transport: T_{ij}
Unit transport cost: $c_{ij}(T_{ij}) = ..$

One large system of equations and inequalities to describe this (GAMS).

Nonlinear Program Model (Monopolist)

- One producer controlling all regions
- Full knowledge of demand system
- Full knowledge of transportation system

$$\begin{aligned} \max_{(D,S,T)\in\mathcal{F}} \quad & \sum_{l\in L} \theta_l(D_l)D_l - \sum_{l\in L} \Psi_l(S_l) - \sum_{i,j} c_{ij}(T_{ij})T_{ij} \\ \text{s.t.} \quad & S_l + \sum_{i,l} T_{il} = D_l + \sum_{l,j} T_{lj}, \quad \forall l \in L \end{aligned}$$

EMP = NLP

2 agents: NLP + VI Model (Monopolist)

- One producer controlling all regions
- Full knowledge of demand system
- Price-taker in transportation system

$$\max_{(D,S,T)\in\mathcal{F}} \quad \sum_{l\in L} \theta_l(D_l)D_l - \sum_{l\in L} \Psi_l(S_l) - \sum_{i,j} \overset{p_{ij}}{\cancel{c_{ij}(T_{ij})}} T_{ij} \tag{1}$$

$$\text{s.t.} \quad S_l + \sum_{i,l} T_{il} = D_l + \sum_{l,j} T_{lj}, \quad \forall l \in L$$

$$p_{ij} = c_{ij}(T_{ij}) \tag{2}$$

empinfo: equilibrium
vi pDef p

EMP = MOPEC $\implies$ MCP

EMP: MOPEC

- Model has the format:

$$\begin{array}{llll} \text{Agent o:} & \min\limits_x & f(x,y) & \\ & \text{s.t.} & g(x,y) \le 0 & (\perp \lambda \ge 0) \\ & & & \\ \text{Agent v:} & H(x,y,\lambda) = 0 & & (\perp y \text{ free}) \end{array}$$

- Difficult to implement correctly (multiple optimization models)
- Can do automatically - simply annotate equations

  empinfo: equilibrium

  min f x defg

  vi H y

  dualvar λ defg
- EMP tool automatically creates an MCP

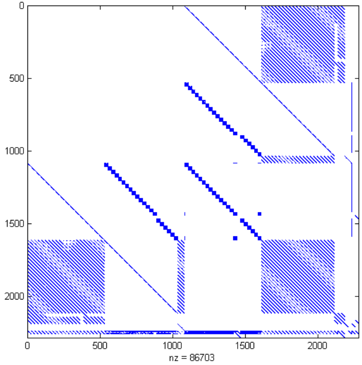
$$\nabla_x f(x,y) + \lambda^T \nabla g(x,y) = 0$$

$$0 \le -g(x,y) \perp \lambda \ge 0$$

$$H(x,y,\lambda) = 0$$

World Bank Project (Uruguay Round)

- 24 regions, 22 commodities
  - Nonlinear complementarity problem
  - Size: 2200 x 2200
- Short term gains $53 billion p.a.
  - Much smaller than previous literature
- Long term gains $188 billion p.a.
  - Number of less developed countries loose in short term
- Unpopular conclusions - forced concessions by World Bank
- Region/commodity structure not apparent to solver



Classic SPE Model (NLP + VI agents)

- One producer controlling all regions
- Price-taker in demand system
- Price-taker in transportation system

$$\max_{(D,S,T)\in\mathcal{F}} \quad \sum_{l\in L} \overset{\pi_l}{\cancel{\theta_l(D_l)}} D_l - \sum_{l\in L} \Psi_l(S_l) - \sum_{i,j} \overset{p_{ij}}{\cancel{c_{ij}(T_{ij})}} T_{ij} \tag{1}$$

$$\text{s.t.} \quad S_l + \sum_{i,l} T_{il} = D_l + \sum_{l,j} T_{lj}, \quad \forall l \in L$$

$$p_{ij} = c_{ij}(T_{ij}) \tag{2}$$

$$\pi_l = \theta_l(D_l) \tag{3}$$

empinfo: equilibrium
vi pDef p
vi thetaDef pi

EMP = MOPEC $\Longrightarrow$ MCP

The Philpott batch problem

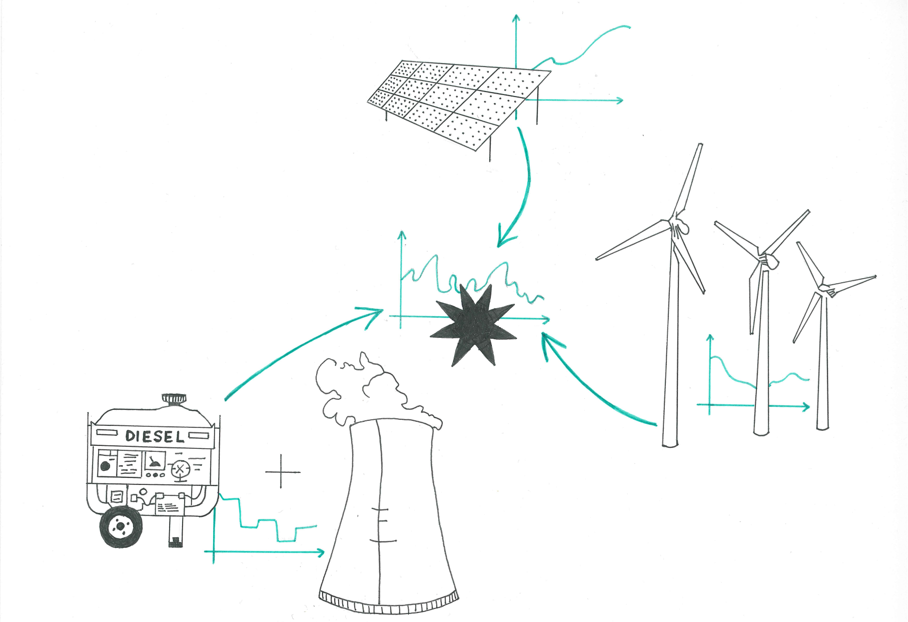
Solar panels:

Battery:

Diesel generator:

Pump storage:

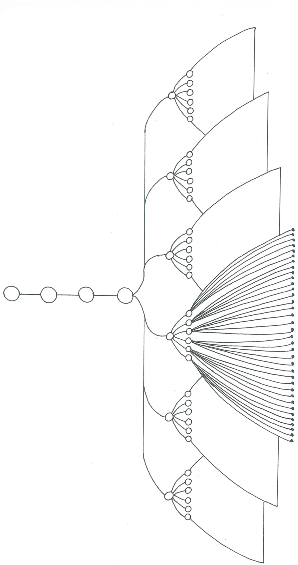
The setup: agents $a =$(solar, wind, diesel, consumer)



Dynamics and uncertainties (risk neutral)

- T stages (use 6 here)
- Scenario tree is data
- Nodes $n \in \mathcal{N}$, n_+ successors
- Stagewise probabilities $\mu(m)$ to move to next stage $m \in n_+$
- Uncertainties (wind flow, cloud cover, rainfall, demand) $\omega_a(n)$
- Actions u_a for each agent (dispatch, curtail, generate, shed), with costs C_a
- State and shared variables (storage, prices)
- Recursive (nested) definition of expected cost-to-go: $\theta(n) = \sum_{m \in n_+} \mu(m) \left(\sum_{a \in \mathcal{A}} C_a(u_a(m)) + \theta(m) \right)$

$t \in 0, 1, 2, 3, 4, 5, 6$



SO Model: the gold standard

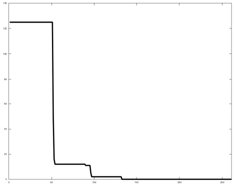
$$\text{SO:} \quad \min_{(\theta,u,x)\in\mathcal{F}(\omega)} \quad \sum_{a\in\mathcal{A}} C_a(u_a(0)) + \theta(0)$$

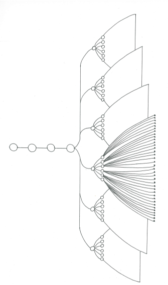
$$\text{s.t.} \quad \theta(n) \geq \sum_{m\in n_+} \mu(m) \left(\sum_{a\in\mathcal{A}} C_a(u_a(m)) + \theta(m) \right)$$

$$\sum_{a\in\mathcal{A}} g_a(u_a(n)) \geq 0$$

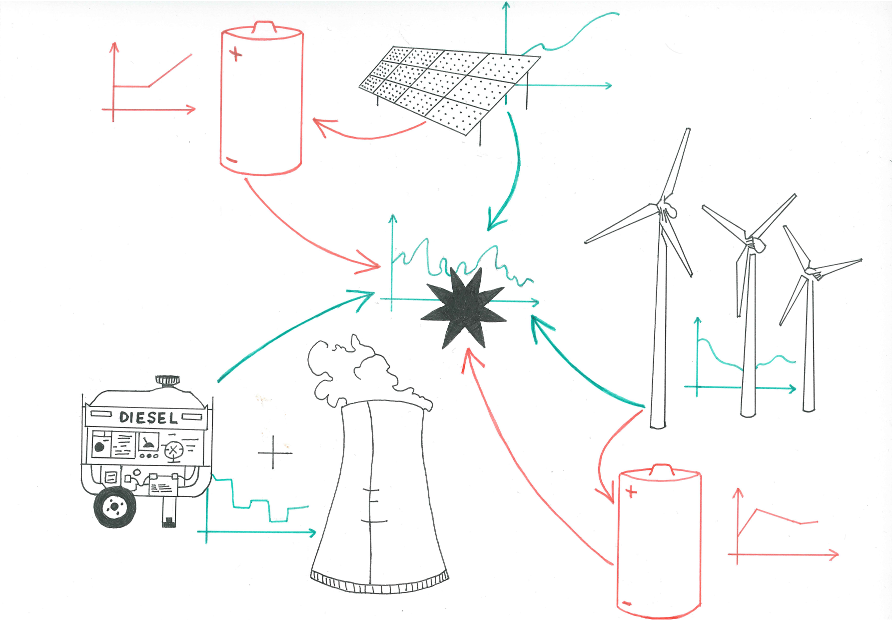
- g_a converts actions into energy.
- Solution (risk neutral, system optimal):
- consumer cost 1,308,201; probability of shortage 19.5%
- No transfer of energy across stages.

Prices π on energy constraint:





Add storage (smoother) to uncertain supply



Add storage

- Storage allows energy to be moved across stages (batteries, pump, compressed air, etc)
- Solution using only battery consumer cost 1,228,357; probability of shortage 11.5%
- Solution using battery and diesel consumer cost 207,476; probability of shortage 1.1%

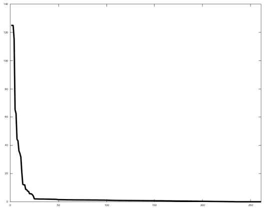
$$\min_{(\theta,u,x)\in\mathcal{F}} \quad \sum_{a\in\mathcal{A}} C_a(u_a(0)) + \theta(0)$$

$$\text{s.t. } x_a(n) = x_a(n_-) - u_a(n) + \omega_a(n)$$

$$\theta(n) \geq \sum_{m\in n_+} \mu(m) \left(\sum_{a\in\mathcal{A}} C_a(u_a(m)) + \theta(m) \right)$$

$$\sum_{a\in\mathcal{A}} g_a(u_a(n)) \geq 0$$

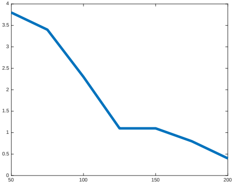
Prices π on energy constraint:



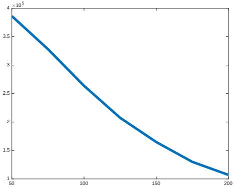
Investment planning: storage/generator capacity

Increasing battery capacity

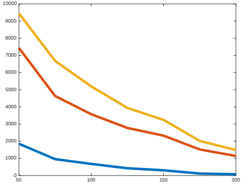
Shortage probability



Consumer cost

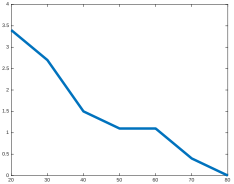


Agent profits

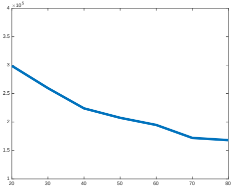


Increasing diesel generator capacity

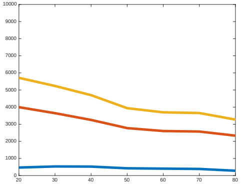
Shortage probability



Consumer cost



Agent profits



Risk averse equilibrium (more details in public lecture)

Replace each agents problem by:

$$\text{RA}(a, \pi, \mathcal{D}_a): \min_{(\theta,u,x)\in\mathcal{F}} \quad Z_a(0) + \theta_a(0)$$

$$\text{s.t. } x_a(n) = x_a(n_-) - u_a(n) + \omega_a(n)$$

$$\theta_a(n) \geq \sum_{m \in n_+} p_a^k(m)(Z_a(m) + \theta_a(m)), \quad k \in K(n)$$

$$Z_a(n) = C_a(u_a(n)) - \pi(n) g_a(u_a(n))$$

- $p_a^k(m)$ are extreme points of the agents risk set $\mathcal{D}_a$ at m
- Constraint: θ_a dominates $\sup_{\mu \in \mathcal{D}_a} \langle \mu, \cdot \rangle$ the dual definition of risk measure
- No longer system optimization, solve using complementarity solver
- Need techniques to treat stochastic optimization problems within equilibrium

Computational results

- Nash equilibrium as a MOPEC

$$\{(\theta_a(n)), u_a(n), n \in \mathcal{N}, x\} \in \arg\min \mathrm{RA}(a, \pi, \mathcal{D}_a)$$

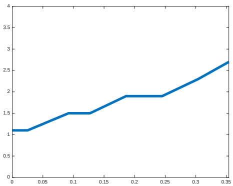
and

$$0 \leq \sum_{a \in \mathcal{A}} g_a(u_a(n)) \perp \pi(n) \geq 0$$

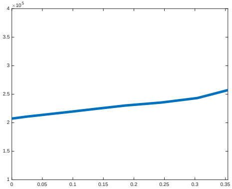
- One optimization per agent, coupled together with solution of complementarity (equilibrium) constraint.

Increasing risk aversion (effect of competition, limited investment)

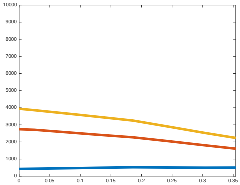
Shortage probability



Consumer cost



Agent profits



What we can do?

- Used in models such as PIES, MERGE, VEMOD, MARKAL, TIMES, KAPSARC, ISEEM, MESSAGE, TEA, TIGER, Gemstone
- Models of Tobin, Nordhaus, Romer
- Frequently used in Computable General Equilibrium (CGE) analyses (GTAP data available), traffic, structural analysis
- Policy analyses such as Uruguay round, NAFTA, USMCA, Brexit
- Equilibrium $\equiv$ complementarity ($\approx$ coupling)
- PATH solver for large scale mixed complementarity problems

$$0 \leq F(x) \perp x \geq 0$$

- Nonsmooth Newton method, efficient linear algebra, available in modeling systems: GAMS, MPSGE, AMPL, AIMMS, Julia, Pyomo

The EMP framework

- Automates all the steps: no need to derive MCP by hand.
- Annotate equations and variables in an `empinfo` file.
- The framework automatically transforms the problem into another computationally more tractable form.

$$
\begin{array}{ll}
\underset{x_i}{\text{minimize}} & f_i(x_i, x_{-i}, \pi), \\
\text{subject to} & g_i(x_i, x_{-i}, \pi) \leq 0, \\
& h_i(x_i, x_{-i}, \pi) = 0, \\
& \text{for } i = 1, \ldots, N, \\
\hline
\pi & \in \mathrm{SOL}(F, K).
\end{array}
$$

```
equilibrium
min f('1') x('1') g('1') h('1')
...
min f('N') x('N') g('N') h('N')
vi F pi K
```

An example of using the EMP framework

- An oligopolistic market equilibrium problem formulated as a NEP:

$$q_i^* \in \quad \mathrm{argmax}_{q_i \geq 0} \quad q_i p \left(\sum_{j=1, j \neq i}^{5} q_j^* + q_i \right) - c_i(q_i), \text{ for } i = 1, \ldots, 5.$$

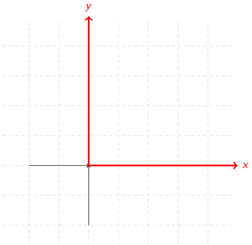
```
variables obj(i); positive variables q(i);
equations defobj(i);
defobj(i)..  obj(i) =E= ...;
model m / defobj /;

file info / '%emp.info%' /;
put info 'equilibrium' /;
loop(i, put 'max', obj(i), q(i), defobj(i) /;);
putclose;
solve m using emp;
```

MIP formulations for Complementarity

Set $y_i = F_i(x)$, then (disjunction)

$$0 \leq y_i, \quad y_i x_i = 0, \quad x_i \geq 0$$



If we know upper bounds on x_i and y_i we can introduce binary variable z_i and model as:

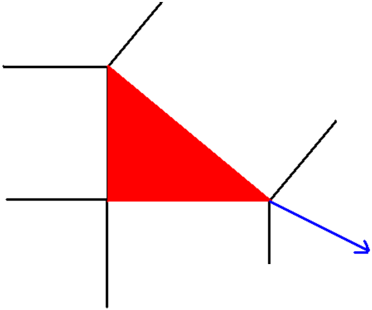
$$0 \leq x_i \leq M z_i, \quad 0 \leq y_i \leq M(1 - z_i)$$

or (without bounds)

$$(x_i, y_i) \in SOS1$$

(or use indicator variables to turn on "fixing" constraints).
Works if bounds are good and problem size is not too large. Issues with bounds on multipliers not being evident. c.f. Optimal topology problems.

Normal map for polyhedral C



projection: $\pi_C(x)$

$$x - \pi_C(x) \in N_C(\pi_C(x))$$

If $-F(\pi_C(x)) = x - \pi_C(x)$ then $z = \pi_C(x)$ solves

$$0 \in F(z) + N_C(z)$$

if and only if we can find x, a zero of the normal map:

$$0 = F(\pi_C(x)) + x - \pi_C(x)$$

Nonsmooth alternatives and approximations (NLPEC)

Alternative: generate generalized derivatives of nonsmooth reformulations

- PATH uses (PC^1) normal map
- Min-map $\min(F_i(x), x_i) = 0$
- Fischer-Burmeister $\Phi(x) = 0$

$$\phi(a,b) = 0 \iff 0 \le a \perp b \ge 0$$

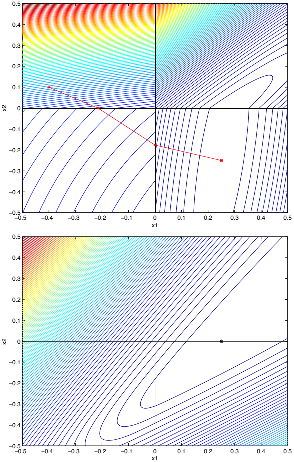
$$\Phi_i(x) \equiv \sqrt{x_i^2 + F_i(x)^2} - x_i - F_i(x)$$

- Smoothing (drive parameter μ to 0)

$$0 = \phi_\mu(F_i(x), x_i), \quad i = 1, 2, \ldots, n$$

$$\phi_\mu(a,b) := \sqrt{a^2 + b^2 + \mu} - a - b$$

- Relaxation $F_i(x)x_i \le \mu$
- Penalization $+\lambda \sum_{i=1}^n F_i(x)x_i$



Extension: Hierarchical models

- Bilevel programs:

$$\begin{array}{ll} \min\limits_{x^*,y^*} & f(x^*,y^*) \\ \text{s.t.} & g(x^*,y^*) \le 0, \\ & y^* \text{ solves } \min\limits_{y} v(x^*,y) \text{ s.t. } h(x^*,y) \le 0 \end{array}$$

- model bilev /deff,defg,defv,defh/;
  empinfo: bilevel min v y defv defh
- EMP tool automatically creates the MPCC

$$\begin{array}{lll} \min\limits_{x^*,y^*,\lambda} & f(x^*,y^*) & \\ \text{s.t.} & g(x^*,y^*) \le 0, & \\ & 0 \le \nabla v(x^*,y^*) + \lambda^T \nabla h(x^*,y^*) & \perp y^* \ge 0 \\ & 0 \le -h(x^*,y^*) & \perp \lambda \ge 0 \end{array}$$

EMP: MPCC: complementarity constraints

$$\begin{array}{ll} \min_{x,y} & f(x,y) \\ \text{s.t.} & g(x,y) \leq 0, \\ & 0 \leq y \perp h(x,y) \geq 0 \end{array}$$

- g, h model "engineering" expertise: finite elements, etc
- $\perp$ models complementarity, disjunctions
- Complementarity "$\perp$" constraints available in AIMMS, AMPL and GAMS
- NLPEC: use the convert tool to automatically reformulate as a parameteric sequence of NLP's
- Solution by repeated use of standard NLP software
  - Problems solvable, local solutions, hard
  - Southern Spars Company (NZ): improved from 5-0 to 5-2 in America's Cup!

MPCC approaches

- Implicit: $\min_x f(x, y(x))$
- Auxiliary variables: $s = h(x, y)$
- NCP functions: $\Phi(y, s) = 0$
- Smoothing: $\Phi_\mu(y, s) = 0$
- Penalization: $\min f(x, y) + \mu y^T s$
- Relaxation: $y^T s \leq \mu$

Different problem classes require different solution techniques

Parametric algorithm: NLPEC

- Reftype mult
- Aggregate none
- Constraint inequality
- Initmu $= 0.01$
- Numsolves $= 5$
- Updatefac $= 0.1$
- Finalmu $= 0$

A solution procedure whereby μ is successively reduced can be implemented as a simple option file to NLPEC

$$\begin{aligned}
\min_{x\in\mathbb{R}^n, y\in\mathbb{R}^m, s\in\mathbb{R}^m} \; & f(x,y) \\
\text{s.t. } & g(x,y) \le 0 \\
& s = h(x,y) \\
& y \ge 0, s \ge 0 \\
& y_i s_i \le \mu, \quad i = 1, \ldots, m.
\end{aligned}$$

Note that a series of approximate problems are produced, parameterized by $\mu > 0$; each of these approximate problems have stronger theoretical properties than the problem with $\mu = 0$

Conclusions

- Equilibrium naturally modeled via complementarity
- Solvers exist for medium to large scale problems
- Frameworks (EMP) exist to streamline model transformations
- empinfo: dualvar, bilevel, equilibrium, vi
- Very large scale models (many agents with many instruments acting strategically) with risk are hard
- Structure exploiting methods can be effective when used carefully
- New algorithms enable solution of more detailed, authentic problems and address underlying policy questions