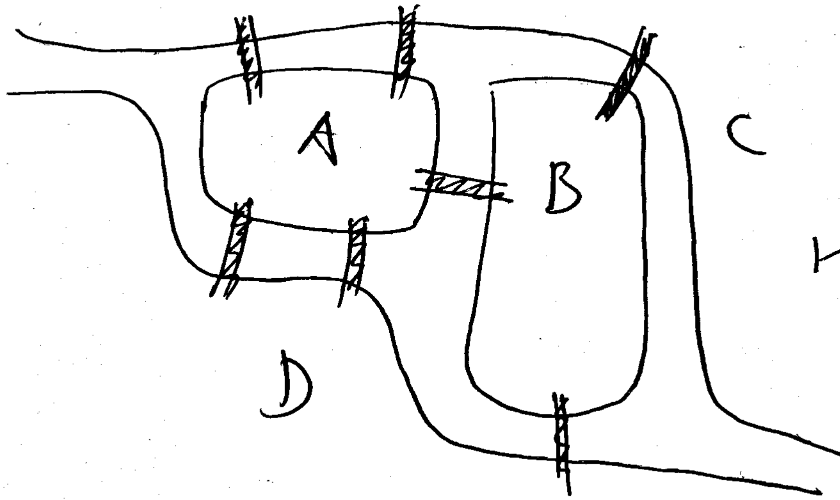
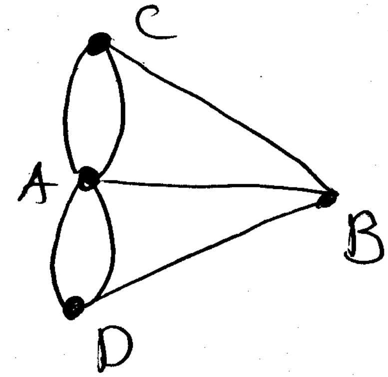
ex The original Königsberg problem cannot be solved positively:

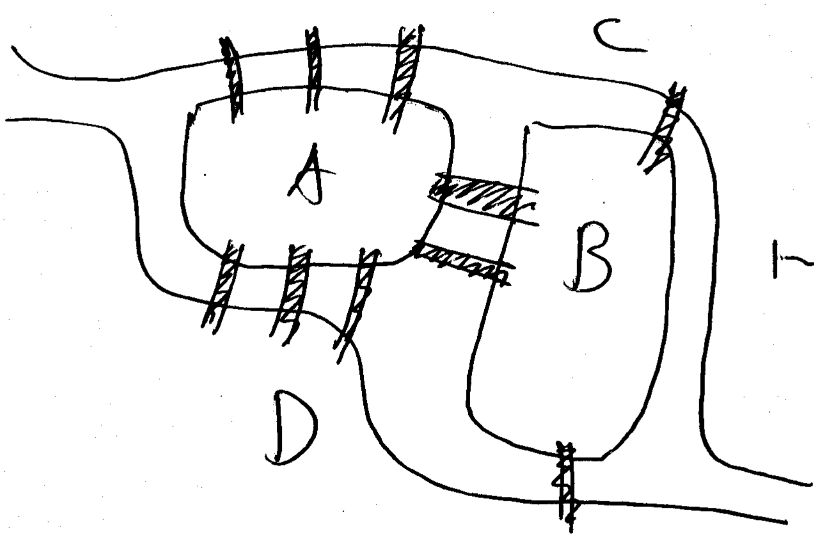


$\mapsto$

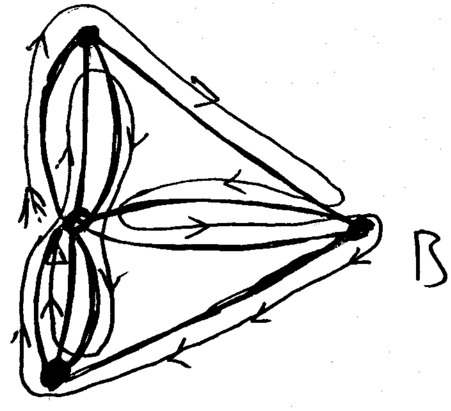


$d(A) = 5 \neq \text{even}!$
(in fact, $d(X) \neq$ even for all X)

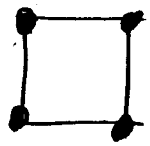
— Adding a few more bridges makes it solvable:



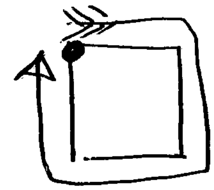
$\mapsto$



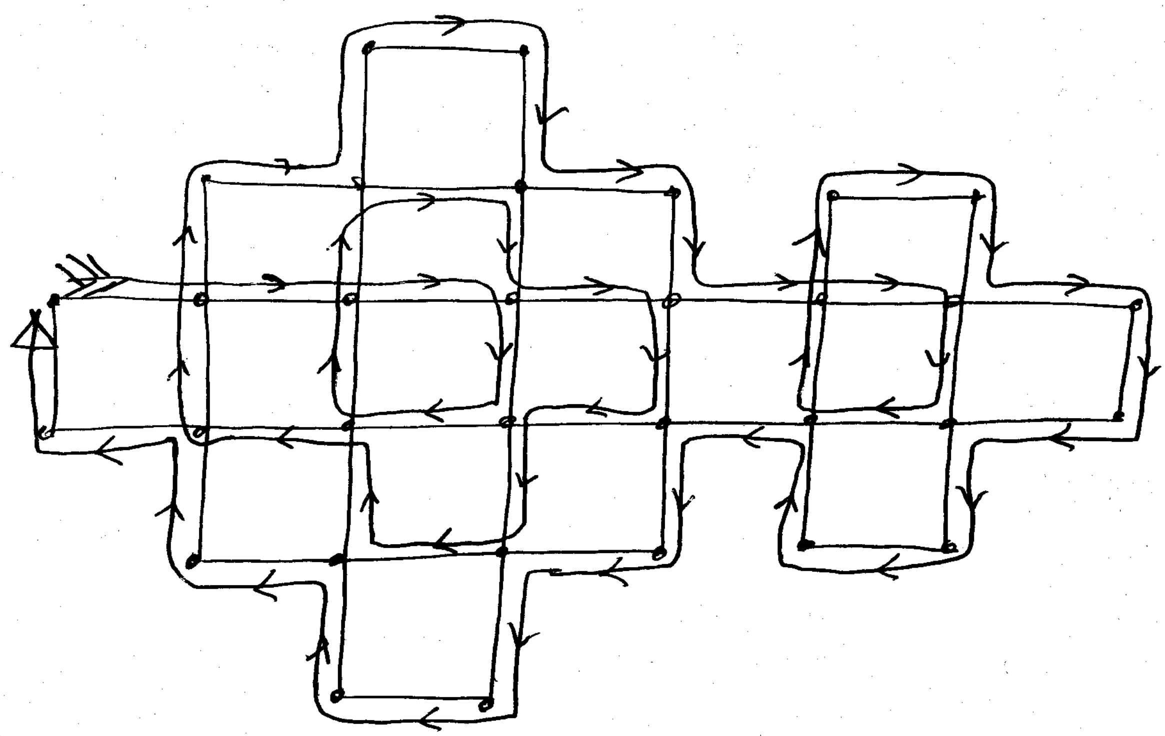
$d(X) = \text{even}$
$\forall X \in \{A, B, C, D\}$!



$G - \text{edges}(E_1)$



E_1'

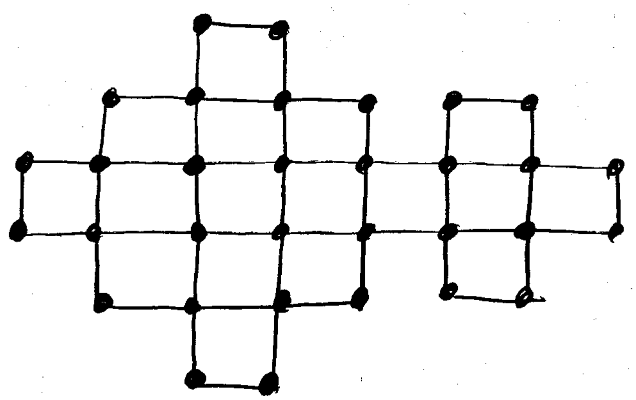


$E_2 = E_1 \cup (E_1')$

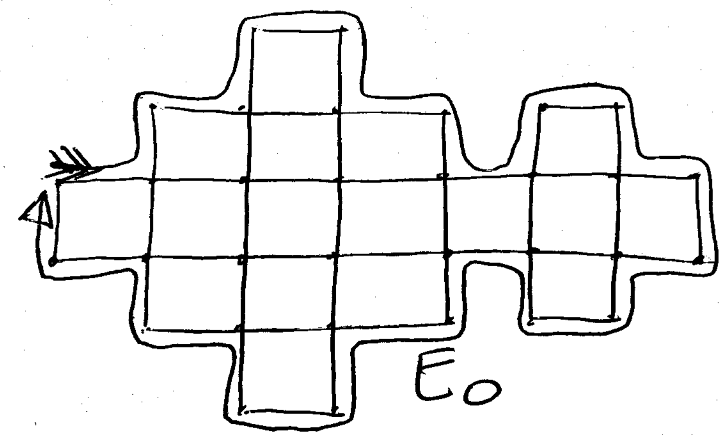
@ this point all the edges of G are covered in E_2, so E_2 cannot be enlarged.

$\Longrightarrow E_2 =$ Eulerian circuit of G!

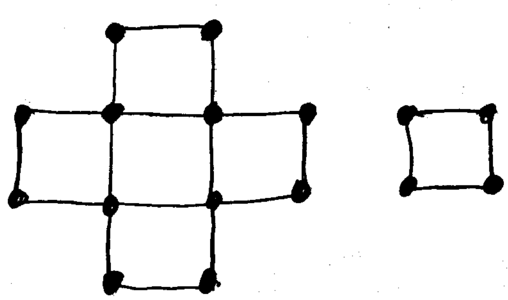
ex



G

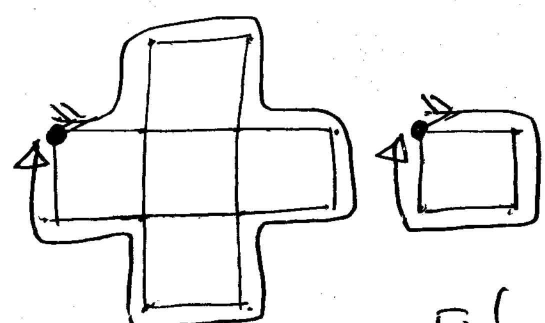


pick some circuit E_0

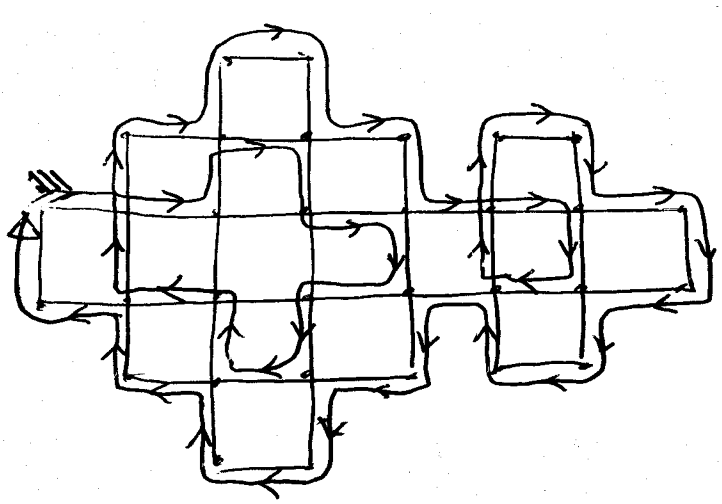


$G' = G - \text{edges}(E_0)$

NB! disregard isolated vertices



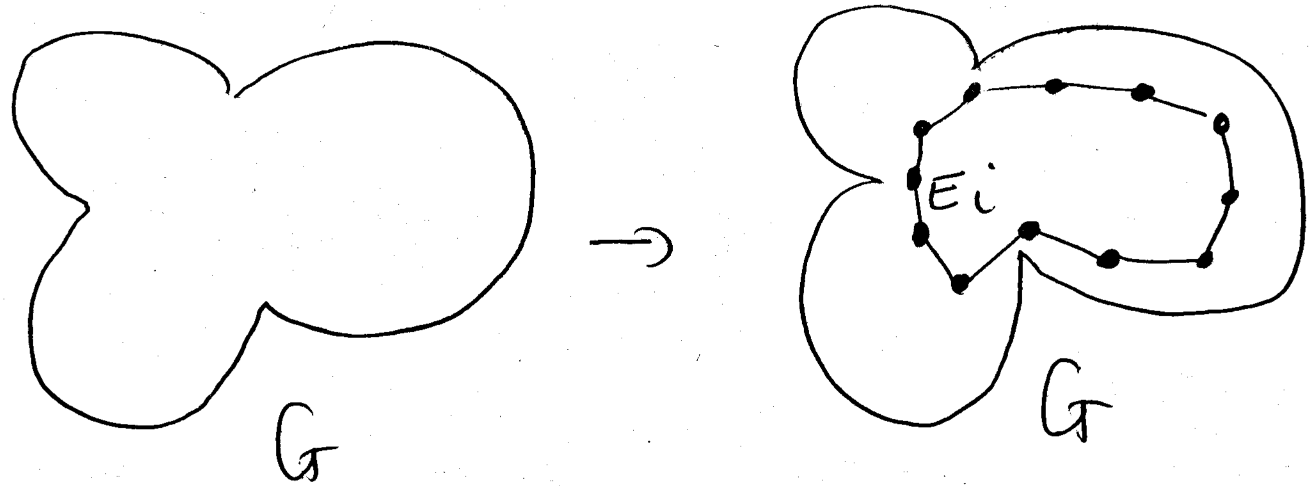
E_1' E_2'

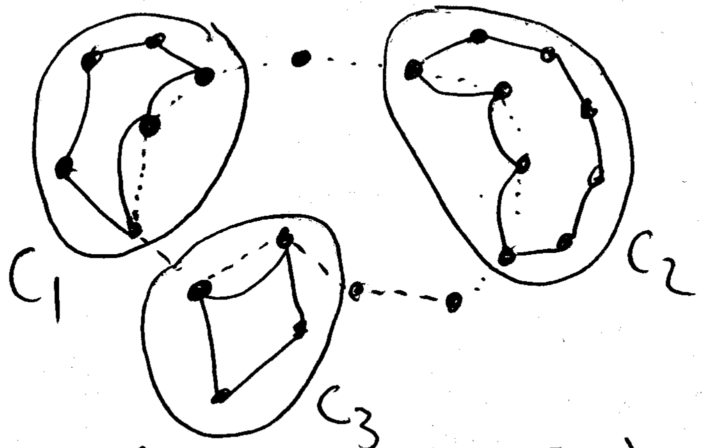


$E_1 = E_0 \cup (E_1' \cup E_2')$

@ this point we have E_1 a circuit strictly larger than E_0, but it is still no an Eulerian circuit!

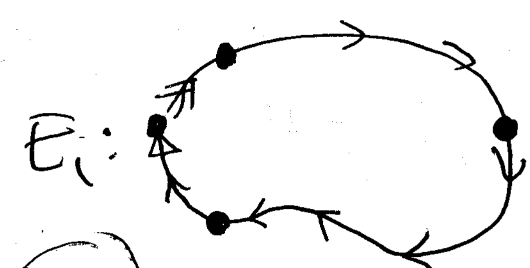
A pictorial presentation:

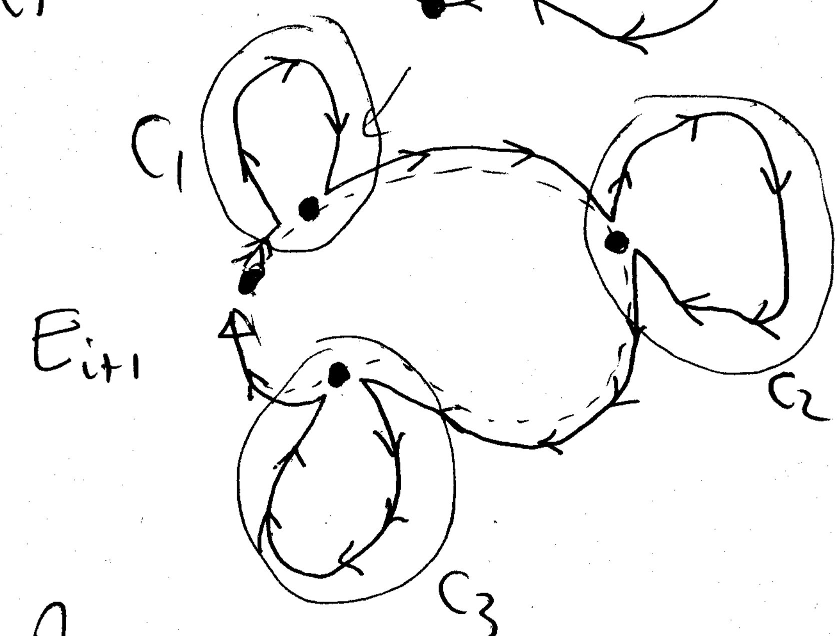




$G' = G - \text{edges}(E_i)$

Now we enlarge E_i to obtain E_{i+1}:





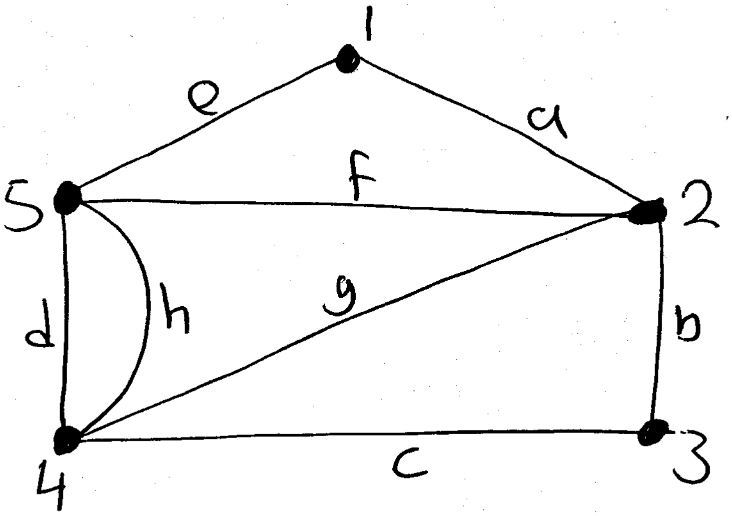
— As long as we can enlarge E_i in this fashion, it is not an Eulerian circuit!

An algorithm to obtain an Eulerian circuit E^* in an Eulerian graph G:

1. Start by picking some circuit E_0
2. While $E_i \neq$ Eulerian circuit of G do:
   - Let $G' = G - \text{edges}(E_i) = C_1 \cup C_2 \cup \ldots \cup C_k$
   - Let $E_1', E_2', \ldots, E_k'$ be circuits in the components $C_1, C_2, \ldots, C_k$ resp. such that for each ℓ the initial & final vertex of E_ℓ' is on E_i
   - Obtain $E_{i+1} = E_i \cup (E_1' \cup \cdots \cup E_k')$ by appending each E_ℓ' to E_i @ the initial vertex of E_ℓ' (that is also in E_i)
   - Goto 2
3. Record $E^* = E_i$ as an Eulerian circuit of G.

Σ

- By labeling the edges of G as well we can present the circuits by a sequence of an initial vertex i and then the edges:



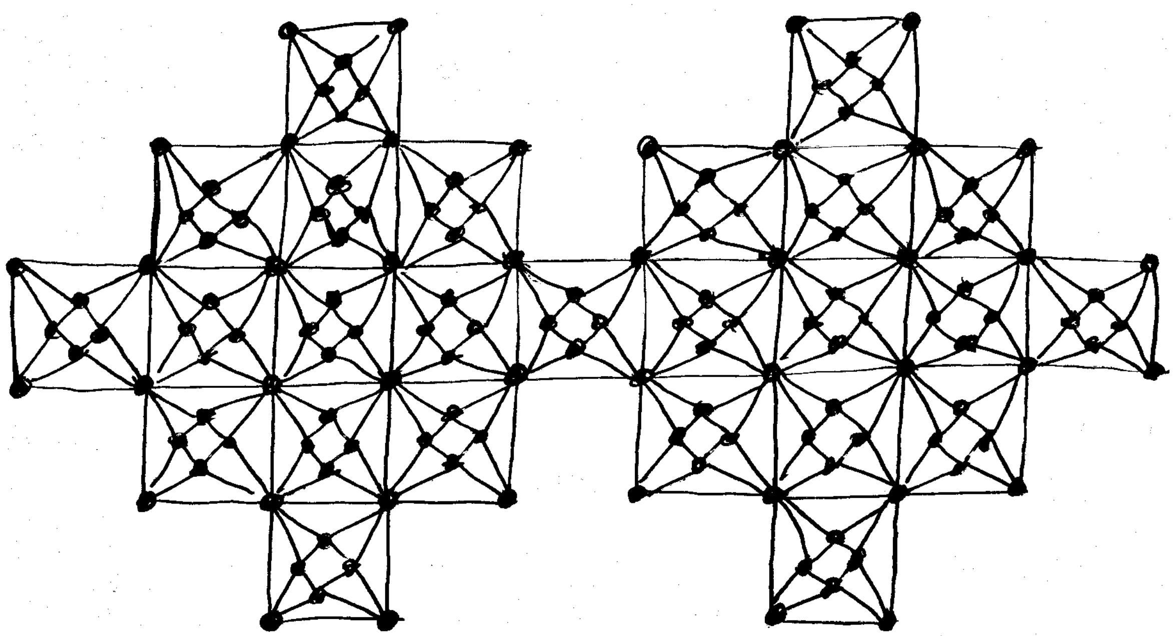
$E_1 = (1, e, h, g, f, d, c, b, a, 1)$

$E_2 = (5, d, c, b, a, e, f, g, h, 5)$

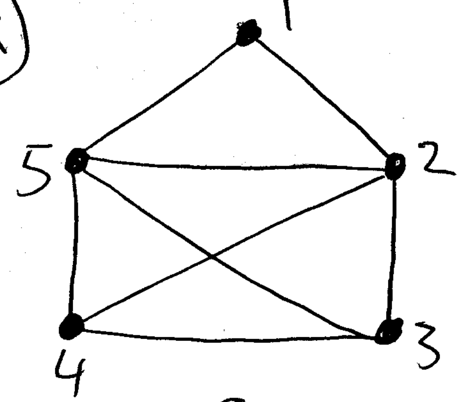
$E_3 = (2, f, h, c, b, g, d, e, a, 2)$

- For a "small" Eulerian graph it is easy to spot Eulerian circuits by trial & error

- But for "large" complex Eulerian graphs that (t & e) is not the way to go:



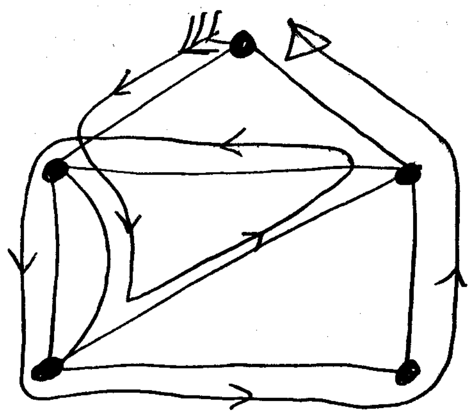
ex

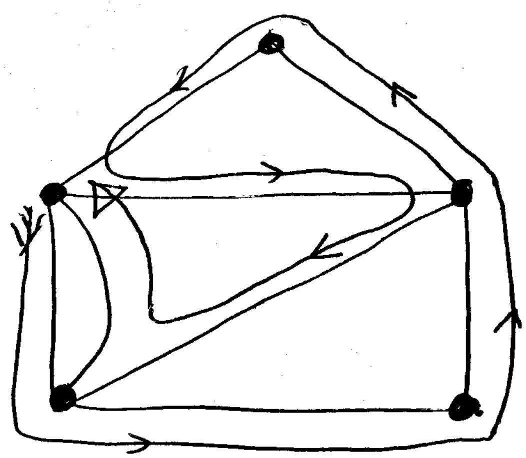


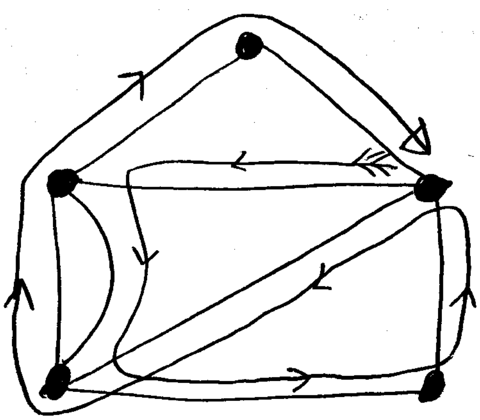
G

Here $d_G(4) = 3 \neq$ even $\Longrightarrow$ G is <u>not</u> Eulerian!

For an Eulerian graph G, there are usually <u>many</u> Eulerian circuits:





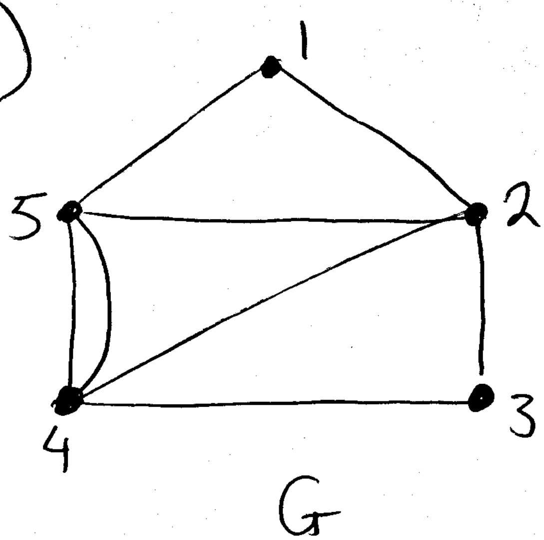


Here we see 3 distinct Eulerian circuits.

$\boxed{\text{Thm}}$ (Euler ~ 1730)

A graph G is Eulerian iff
- G is connected &
- $d_G(u)$ is even for each $u \in V$ the vertex set of G.

(ex)

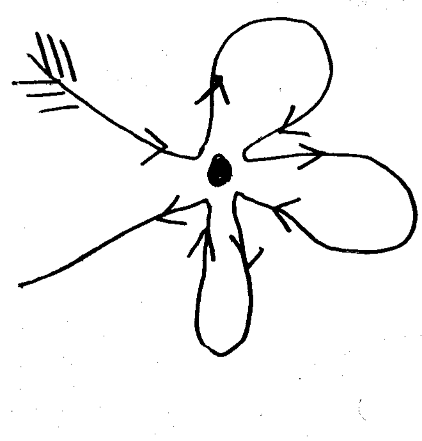


$d_G(1) = d_G(3) = 2$

$d_G(2) = d_G(4)$
$= d_G(5)$
$= 4$

all degrees are even
$\Longrightarrow$ G is Eulerian!

— To see necessity:



If G has an Eulerian circuit, then we go "into" each vertex equally as often as we "leave" it

$\Longrightarrow$ $d_G(u)$ must be even for all u!

Def A graph that has an Eulerian circuit is called an <u>Eulerian graph</u>.

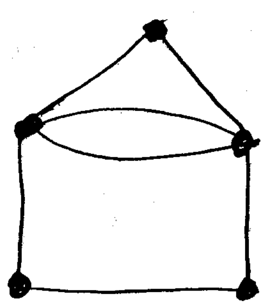
<u>NB</u>: An Eulerian graph must be connected!

— How do we determine whether a graph is Eulerian or not?

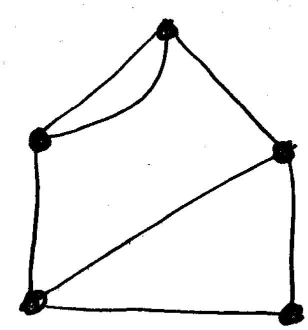
— If the graph is Eulerian, how do we find an Eulerian circuit?

ex

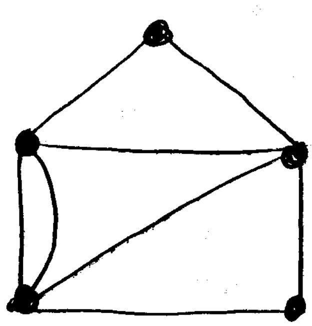
G_1



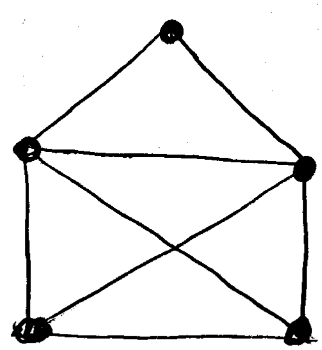
G_2



G_3



G_4



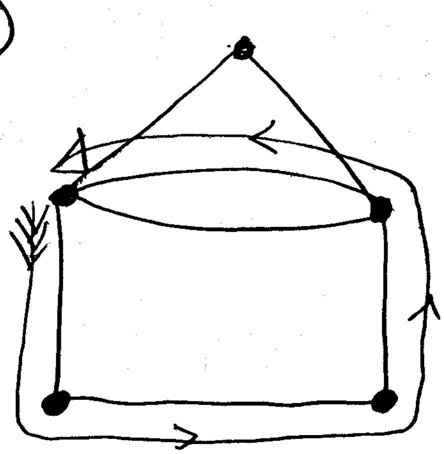
G_1 & G_3 = Eulerian;
G_2 & G_4 are not.

Ch-10 Circuits

- Recall that a circuit is a closed trail where the edges are distinct.

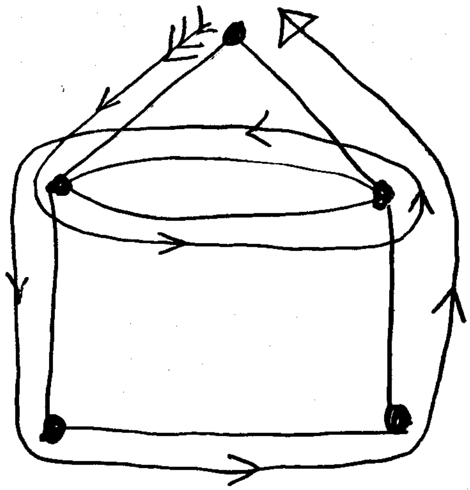
Def A circuit in a graph that contains every edge of the graph is called an Eulerian circuit.

ex



- A circuit (in fact a cycle!)
- not Eulerian

ex



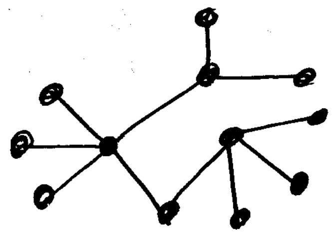
- An Eulerian circuit (not a cycle!)

Def

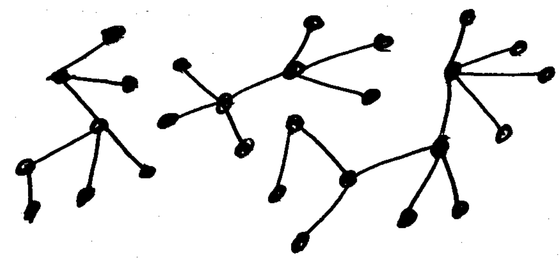
i) A simple connected graph with no cycles in it is a *tree*.

ii) A simple graph in which each component is a tree is called a *forest*.

ex



a tree



a forest
(w/ 3 components)

Fact

- A tree on n vertices has $n-1$ edges.
- A forest on n vertices & with k components has $n-k$ edges.