A Note on Strongly Simplicial Vertices of Powers of Trees

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Abstract

For a tree T and an integer $k \geq 1$, it is well known that the k-th power T^k of T is strongly chordal and hence has a strong elimination ordering of its vertices. In this note we obtain a complete characterization of strongly simplicial vertices of T^k, thereby characterizing all strong elimination orderings of the vertices of T^k.

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1 Introduction

Strongly chordal graphs have received much attention since first defined in [9], in particular because they yield polynomial time solvability of the domatic set and the domatic partition problems. For more information on these problems we refer the reader to [4], [10], [11] and [6]. In [9] a characterization of strongly chordal graphs in terms of balanced matrices is given. In [13] and [8] it is shown that any power of a strongly chordal graph is again strongly chordal. We should note that this does not hold for chordal graphs in general: Only odd powers of chordal graphs are guaranteed to be again chordal, as was first explicitly shown in [3]. A simpler proof of this result can be found in [2]. Since a tree is strongly chordal, then any power of a tree is again strongly chordal. Although a direct consequence of [13] and [8], this was also shown explicitly in [7]. Since any power of a tree is strongly chordal, this implies that it has a strong elimination orderings of its vertices.

The purpose of this note is to characterize completely the strongly simplicial vertices of a power of a tree, and thereby give a complete characterization of the strongly elimination orderings of the vertices of a power of a tree. Strongly simplicial vertices of powers of trees can be applied to obtain an optimal greedy vertex coloring of squares of outerplanar graphs, as described in detail in [1]. In particular, we derive here Theorem 2.9 and Corollary 2.11, the latter of which was used to obtain the mentioned optimal greedy coloring algorithm in [1]. We should stress that the main result of this note is not a theorem, but rather Definition 2.2. Namely the mere statement of what we mean by k-strong simplicity of a vertex in a tree, as described in Lemma 2.1.

General notation The set $\{1, 2, 3, \ldots\}$ of natural numbers will be denoted by $\mathbb{N}$. All graphs in this note are assumed to be simple and undirected unless otherwise stated. The degree of a vertex

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u in graph G is denoted by $d_G(u)$. We denote by $N_G(u)$ the *open neighborhood* of u in G, that is the set of all neighbors of u in G, and by $N_G[u]$ the *closed neighborhood* of u in G, that additionally includes u. The distance $\partial_G(u,v)$, between vertices u and v, is the number of edges in a shortest path between them. When the graph in question is clear from context, we omit the subscript in the notation. For a graph G and $k \in \mathbb{N}$, the *power graph* G^k is the simple graph with the same vertex set as G, but where every pair of vertices of distance k or less in G is connected by an edge. In particular, G^2 is the graph in which, in addition to edges of G, every two vertices with a common neighbor in G are also connected with an edge. The closed neighborhood of a vertex u in G^k will be denoted by $N_G^k[u]$ and the degree of vertex u will be denoted by $d_k(u)$.

Recall the following definition:

Definition 1.1 *A vertex u in a graph G is* simplicial *if $N_G[u]$ induces a clique in G. If u is simplicial and $\{N_G[v] : v \in N_G[u]\}$ is linearly ordered by set inclusion, then u is* strongly simplicial.

A graph G is *strongly chordal* if it has a *strong elimination ordering* of its vertices $V(G) = \{u_1, \ldots, u_n\}$, such that each vertex u_i is strongly simplicial in the subgraph of G induced by u_i and the previous vertices $u_1, \ldots, u_{i-1}$. Clearly, a vertex of a tree is strongly simplicial if, and only if, it is a leaf, which gives us a complete description of when exactly an ordering is a strong elimination ordering of the tree. To describe strong simplicity in T^k for arbitrary $k \in \mathbb{N}$, we need to introduce some special notation for trees.

Notation and terminology of trees The leaves of a tree T will be denoted by $L(T)$. The *diameter* of T is the number of edges in a longest path in T and will be denoted by $\mathrm{diam}(T)$. For a tree T with $\mathrm{diam}(T) \geq 1$ we can form the *pruned* tree $\mathrm{pr}(T) = T - L(T)$. For two vertices u and v of a tree T, the unique path between them will be denoted by $p_T(u,v)$ or by $p(u,v)$ when there is no danger of ambiguity. The vertices of this path, including both u and v, is given by $V(p(u,v))$. A *center* of T is a vertex of distance at most $\lceil \mathrm{diam}(T)/2 \rceil$ from all other vertices of T. A center of T is either unique or one of two unique adjacent vertices. Clearly, the power graph T^k of a tree T is only interesting when $k \in \{1, \ldots, \mathrm{diam}(T)-1\}$. For $U \subseteq V(T)$ the *join* of U in T is the unique smallest subtree of T connecting all of U together. The *connector* of three leaves of T is the unique vertex of degree three in the join of the leaves.

Let T be rooted at $r \in V(T)$. The *k-th ancestor* of $u \in V(T)$, if it exists, is the vertex on $p(u,r)$ of distance k from u, and is denoted by $a_r^k(u)$. An *ancestor* of u is a vertex of the form $a_r^k(u)$ for some $k \geq 0$. Note that u is viewed as an ancestor of itself. The *descendants* of u, denoted by $D_r[u]$, is the collection of all the vertices having u as an ancestor. For $u \in V(T)$, the distance $\partial_T(u,r)$ to the root r will be referred to as the *level* of u and denoted by $l_T(u)$ or by $l(u)$ when there is no danger of ambiguity. For $U \subseteq V(T)$ the *least common ancestor* of U, denoted by $\mathrm{lca}(U)$, is the unique common ancestor of U on the largest level. For a vertex u in T the subset $R_u \subseteq V(T)$ contains all $r \in V(T)$ with $\partial_T(u,r)$ maximum. Note that this definition of R_u works for unrooted trees T. Also note that for each $u \in V(T)$ we necessarily have $R_u \subseteq L(T)$. In fact, we have the following observations that will be useful in the next section.

Claim 1.2 *Let T be a tree and $u \in V(T)$. Then the center(s) of T is(are) on the path $p(u,r)$ for each $r \in R_u$.*

Proof. Root T at a center c. If $r \in V(T)$ is such that c is not on the path $p(u,r)$ then there is a longer path going from u through c, and hence $r \notin R_u$. □

Claim 1.3 *Let T be a tree, $u \in V(T)$ and $r \in R_u$. Then r is an endvertex of a path of T of maximum length* $\mathrm{diam}(T)$*.*

Proof. By Claim 1.2 the center(s) of T is(are) on $p(u, r)$. If c is a center of T, then there is a vertex x that is (i) an endvertex of a maximum length path of T and (ii) such that $c \in V(p(u, x))$. By definition we have $\partial_T(u, x) \le \partial_T(u, r)$ and hence $\partial_T(c, x) \le \partial_T(c, r)$. Therefore equality holds and r is also an endvertex of a maximum length path in T. □

By the above two Claims 1.2 and 1.3 we have the following.

Claim 1.4 *Let T be a tree with* $\mathrm{diam}(T) = d$ *and $u \in V(T)$. If $u \in V(T)$ and $r \in R_u$, then* $\partial_T(u, r) \ge \lceil d/2 \rceil$*.*

With this setup we can start to discuss our first results, the characterization of simplicial vertices of powers of trees.

2 Characterization of simplicial vertices

We start with the following lemma.

Lemma 2.1 *Let T be a tree, $u, r \in V(T)$ and $k \in \mathbb{N}$. If T is rooted at r, let $\mathbf{P}_{u;k}(r)$ be the following statement:*

> $\mathbf{P}_{u;k}(r)$ *: If $a_r^k(u)$ exists, then $\partial_T(v, a_r^{k-1}(u)) \le k-1$ for all $v \in D_r[a_r^{k-1}(u)]$. Otherwise (if $a_r^k(u)$ does not exist), T^k is a complete graph.*

Then the truth value of $\mathbf{P}_{u;k}(r)$ is independent on $r \in R_u$.

Proof. By definition of R_u the k-th ancestor $a_r^k(u)$ exists for one particular $r \in R_u$ iff $a_r^k(u)$ exists for all $r \in R_u$. Hence, we can assume that $a_r^k(u)$ exists for all $r \in R_u$. In this case we may further assume u to be a leaf of T, since otherwise $\mathbf{P}_{u;k}(r)$ is false regardless of $r \in R_u$. For $r, r' \in R_u$ it suffices to show that if $\mathbf{P}_{u;k}(r)$ does not hold, then $\mathbf{P}_{u;k}(r')$ does not hold either:

Since all three vertices r, r' and u are leaves of T, we have the connector $u' \in V(T) \setminus \{u, r, r'\}$. Clearly we have $\partial_T(u', r) = \partial_T(u', r')$. Looking at the join of r, r' and u, there are two cases to consider.

If $a_r^{k-1}(u) \in V(p(u, u'))$, then $a_r^{k-1}(u) = a_{r'}^{k-1}(u)$ and $D_r[a_r^{k-1}(u)] = D_{r'}[a_{r'}^{k-1}(u)]$ so $\mathbf{P}_{u;k}(r')$ is also false in this case.

If $a_r^{k-1}(u) \in V(p(u', r))$, then $a_{r'}^{k-1}(u)$ is the corresponding vertex of $V(p(u', r'))$ at the same distance from u' as $a_r^{k-1}(u)$ is. Assume there is a descendant v of $a_r^{k-1}(u)$ with $\partial_T(v, a_r^{k-1}(u)) \ge k$. We must consider three separate cases of the location of the connector v' of r, r' and v to be.

First case, $v' \in V(p(u, u'))$: Here $v \in D_r[a_r^{k-1}(u)] \cap D_{r'}[a_{r'}^{k-1}(u)]$ and further $\partial_T(v, a_{r'}^{k-1}(u)) = \partial_T(v, a_r^{k-1}(u)) \ge k$, so we have $\mathbf{P}_{u;k}(r')$ to be false in this case.

Second case, $v' \in V(p(u', r))$: Here $\partial_T(v', a_r^{k-1}(u)) \le \partial_T(v', a_{r'}^{k-1}(u))$ and hence

$$\begin{aligned}
\partial_T(v, a_{r'}^{k-1}(u)) &= \partial_T(v, v') + \partial_T(v', a_{r'}^{k-1}(u)) \\
&\ge \partial_T(v, v') + \partial_T(v', a_r^{k-1}(u)) \\
&= \partial_T(v, a_r^{k-1}(u)) \\
&\ge k.
\end{aligned}$$

Since $a_{r'}^{k-1}(u) \in V(p(v,r'))$ the vertex v is a descendant of $a_{r'}^{k-1}(u)$ when T is rooted at r'. Hence $\mathbf{P}_{u;k}(r')$ is false in this case as well.

THIRD CASE, $v' \in V(p(u',r'))$: By definition of R_u we have that $\partial_T(u,r') \geq \partial_T(u,v)$, and hence $\partial_T(v',r') \geq \partial_T(v',v)$. By symmetry we have therefore

$$\begin{aligned}
\partial_T(r, a_{r'}^{k-1}(u)) &= \partial_T(r', a_r^{k-1}(u)) \\
&= \partial_T(r',v') + \partial_T(v', a_r^{k-1}(u)) \\
&\geq \partial_T(v,v') + \partial_T(v', a_r^{k-1}(u)) \\
&= \partial_T(v, a_r^{k-1}(u)) \\
&\geq k.
\end{aligned}$$

Since $a_{r'}^{k-1}(u) \in V(p(r,r'))$ the vertex r is a descendant of $a_{r'}^{k-1}(u)$ when T is rooted at r'. Hence $\mathbf{P}_{u;k}(r')$ is false in this final case. This completes our proof of the lemma. □

REMARK: The statement $\mathbf{P}_{u;k}(r)$ can, at first sight, seem complex. By the right view point it is however quite natural and simple:

1. First root T at the vertex u.
2. The vertices at the lowest level constitute the set $R_u \subseteq L(T)$.
3. Pick $r \in R_u$ and re-root T at r.
4. If $a_r^k(u)$ does not exist, then $\mathbf{P}_{u;k}(r)$ is true only if $k \geq \mathrm{diam}(T)$.
5. If $a_r^k(u)$ does exist, then $\mathbf{P}_{u;k}(r)$ is true only if u is on the lowest level of the sub-tree of T that is rooted at $a_r^{k-1}(u)$.

By Lemma 2.1 the following definition makes sense.

Definition 2.2 *Let T be a tree and $k \in \mathbb{N}$. We say that a vertex $u \in V(T)$ is* k-strongly simple*, or* k-ss *for short, if the statement $\mathbf{P}_{u;k}(r)$ from Lemma 2.1 is true for one (and hence all) $r \in R_u$.*

Note that by definition we have $a_r^0(u) = u$ for every vertex $u \in V(T)$. Hence, a vertex u in a proper tree T (with at least one edge) is 1-ss in T if, and only if, u is a leaf of T. Also note that in general for $1 \leq k < \mathrm{diam}(T)$, only leaves of T can possibly be k-ss.

By Claim 1.4 and Definition 2.2 we have the following.

Corollary 2.3 *If $1 \leq k' \leq k \leq \lceil d/2 \rceil$ and $u \in V(T)$ is k-ss, then u is also k'-ss.*

The next result describes k-strong simplicity for the remaining interesting values of k:

Theorem 2.4 *Let T be a tree and $u \in V(T)$. If $d = \mathrm{diam}(T)$ and $k \in \{\lceil d/2 \rceil, \ldots, d-1\}$, then the following are equivalent:*

1. *u is an endvertex of a path of maximum length d.*
2. *u is k-ss.*
3. *u is $\lceil d/2 \rceil$-ss.*

Proof. By Definition 2.2 we clearly have (1) $\Rightarrow$ (2) $\Rightarrow$ (3). What remains to show is (3) $\Rightarrow$ (1).

Let $h = \lceil d/2 \rceil$ and assume that u is h-ss in T. Since $a_r^h(u)$ exists for any $r \in R_u$ we have $\partial_T(v, a_r^{h-1}(u)) \leq h-1$ for all $v \in D_r[a_r^{h-1}(u)]$. By Claim 1.2 the center(s) is(are) contained in $V(p(u,r))$ and by the value of h, also contained in $D_r[a_r^h(u)]$. Let c be the center that is closest to u:

If $c \in D_r[a_r^{h-1}(u)]$, then $D_r[c] \subseteq D_r[a_r^{h-1}(u)]$ and hence, in particular, $\partial_T(v, a_r^{h-1}(u)) \leq h-1$ for all $v \in D_r[c]$. Since $D_r[c]$ contains a vertex x such that $\partial_T(x,r) = d$ and hence $\partial_T(x,c) = \lfloor d/2 \rfloor$, we must have $h - 1 \geq \partial_T(x, a_r^{h-1}(u)) = \partial_T(x,c) + \partial_T(c, a_r^{h-1}(u)) = \lfloor d/2 \rfloor + \partial_T(c, a_r^{h-1}(u))$, which can only occur if d is odd and $a_r^{h-1}(u) = c$. In this case $\partial_T(u,c) = \lfloor d/2 \rfloor$ and hence u is an endvertex of maximum length path of T.

If $c \in D_r[a_r^h(u)] \setminus D_r[a_r^{h-1}(u)]$, then $c = a_r^h(u)$ and d must be even. Therefore $h = \partial_T(u, a_r^h(u)) = \partial_T(u,c)$ and u is an endvertex of a maximum length path of T in this case as well. This completes the proof. □

Remarks: What is defined to be an "extreme leaf" of T by Kearney and Corneil in [7] is precisely a vertex that satisfies one condition in Theorem 2.4 (and hence all of them), that is a k-ss vertex of T where $k \in \mathbb{N}$ and $d/2 \leq k < d$.

The following is the first step toward a complete description of strongly simplicial vertices of powers of trees:

Theorem 2.5 *Let T be a tree and $k \in \mathbb{N}$. A k-ss vertex of T is strongly simplicial in the power graph T^k.*

To prove Theorem 2.5, we will use the following from [2, Lemma 2.2, p. 45]:

Lemma 2.6 *If T is a tree rooted at $r \in V(T)$ and $u \in V(T)$, then all the vertices of T on levels at most $l(u)$ and of distance at most k from u, form a clique in T^k.*

Proof. (Theorem 2.5:) We may assume $k < \mathrm{diam}(T)$. Let u be a k-ss vertex of T and let $r \in R_u$ be a fixed root. By Lemma 2.6 $N_T^k[u]$ forms a clique in T^k.

We now show that $\{N_T^k[v] : v \in N_T^k[u]\}$ is linearly ordered by set inclusion. For $u', u'' \in N_T^k[u]$ we show that if $l(u') \geq l(u'')$ then $N_T^k[u'] \subseteq N_T^k[u'']$. Assume $l(u') \geq l(u'')$ and let $v \in N_T^k[u']$. If $\mathrm{lca}(u',u'')$ is a descendant of $\mathrm{lca}(u',v)$, then $\mathrm{lca}(u',u'') \in V(p(u',v))$ and hence $\partial_T(u'',v) \leq \partial_T(u',v) \leq k$ so $v \in N_T^k[u'']$. Otherwise $\mathrm{lca}(u',v)$ must be a descendant of $\mathrm{lca}(u',u'')$. Here we further consider two cases, depending on where u is: If $\mathrm{lca}(u',u'')$ is a descendant of $\mathrm{lca}(u,u'')$, then, by the k-strong simplicity of u, we have $l(v) \leq l(u)$ and hence $\partial_T(u'',v) \leq \partial_T(u'',u) \leq k$ and hence $v \in N_T^k[u'']$. Otherwise, in this case, $\mathrm{lca}(u,u'')$ is a descendant of $\mathrm{lca}(u',u'')$ and hence $\mathrm{lca}(u,u'') \in V(p(u'', \mathrm{lca}(u',u'')))$. Since $l(u') \geq l(u'')$ and $l(u) \geq l(v)$ we have

$$\begin{aligned}
\partial_T(u'',v) &= \partial_T(u'', \mathrm{lca}(u'',u')) + \partial_T(\mathrm{lca}(u'',u'),v) \\
&\leq \partial_T(u', \mathrm{lca}(u'',u')) + \partial_T(\mathrm{lca}(u'',u'),u) \\
&= \partial_T(u',u), \\
&\leq k,
\end{aligned}$$

showing that $v \in N_T^k[u'']$ in this final case. This completes the proof that $\{N_T^k[v] : v \in N_T^k[u]\}$ is linearly ordered by set inclusion. □

Lemma 2.7 *Let T be a tree, $u \in V(T)$ and T be rooted at $r \in R_u$. If $a_r^k(u)$ exists and there is a descendant w of $a_r^{k-1}(u)$ of distance k or more from $a_r^{k-1}(u)$, then u is not strongly simplicial in T^k.*

Proof. We may assume that $l(w) = l(u) + 1$. Going upward from w toward the root r, let v be the first ancestor of w that is contained in $N_T^k[u]$ (such a vertex exists, since w is a descendant of $a_r^{k-1}(u) \in N_T^k[u]$.) Since $\partial_T(w, v) \leq \partial_T(w, a_r^{k-1}(u)) = k$ we have

$$w \in N_T^k[v] \setminus N_T^k[a_r^k(u)]. \tag{1}$$

If $\partial_T(u, w) \leq k$, then $N_T^k[u]$ is not a clique in T^k since $\partial_T(w, a_r^k(u)) = k+1$, and hence u is not even simplicial. So we assume $\partial_T(u, w) > k$. In this case, since $v \in V(p(u, w))$, we have by definition of $r \in R_u$ that $\partial_T(w, v) \leq \partial_T(a_r^k(u), r)$, and hence there is a unique vertex w' on the path $p(a_r^k(u), r)$ with $\partial_T(w, v) = \partial_T(a_r^k(u), w')$. Therefore we have

$$\begin{aligned}
\partial_T(v, w') &= \partial_T(v, a_r^k(u)) + \partial_T(a_r^k(u), w') \\
&= \partial_T(w, v) + \partial_T(v, a_r^k(u)) \\
&= \partial_T(w, a_r^k(u)) \\
&= k + 1.
\end{aligned}$$

But since we also have $\partial_T(w', a_r^k(u)) = \partial_T(w, v) \leq \partial_T(w, a_r^{k-1}(u)) = k$, then $w' \in N_T^k[a_r^k(u)] \setminus N_T^k[v]$, which together with (1) shows that u is not strongly simplicial in T^k. □

Let T be a tree, $u \in V(T)$ and T be rooted at $r \in R_u$. If $a_r^k(u)$ does not exist, then $N_T^k[u] = V(T)$. If $k < \mathrm{diam}(T)$ then $N_T^k[u]$ cannot induce a clique in T and hence u is not simplicial, let alone strongly simplicial in T. We summarize:

Lemma 2.8 *If $a_r^k(u)$ does not exist and $k < \mathrm{diam}(T)$, then u is not strongly simplicial in T^k.*

By Theorem 2.5 and Lemmas 2.7 and 2.8 we have the following.

Theorem 2.9 *For a tree T and $k \in \mathbb{N}$, the vertex $u \in V(T)$ is k-ss in T if, and only if, u is strongly simplicial in T^k.*

Note: Theorem 2.9 yields a procedure to characterize all strong elimination orderings $u_1, \ldots, u_n$ of a power of a tree T on n vertices, and hence an algorithm to list them all in polynomial time.

By Theorems 2.4 and 2.9 we have the following corollary:

Corollary 2.10 *Let T be a tree and $u \in V(T)$. If $d = \mathrm{diam}(T)$ and $k \in \{\lceil d/2 \rceil, \ldots, d-1\}$, then u is strongly simplicial in T^k if, and only if, u is strongly simplicial in $T^{\lceil d/2 \rceil}$.*

For a tree T we can recursively define $T^{(i)}$ by $T^{(0)} = T$ and $T^{(i)} = \mathrm{pr}(T^{(i-1)})$ for $i \in \mathbb{N}$, as long as $T^{(i-1)}$ has leaves, that is, is neither empty nor one vertex. With this notation we obtain the following.

Corollary 2.11 *Let T be a tree with $\mathrm{diam}(T) = d \geq 2$. For $u \in V(T)$ and $k \in \{1, \ldots, \lceil (d-1)/2 \rceil\}$, the following are equivalent:*

1. *For a center $c \in V(T)$, the vertex $a_c^{k-1}(u)$ is a leaf of $T^{(k-1)}$.*

2. u is strongly simplicial in T^k.

Proof. Assume $\partial_T(u,c) \leq k-1$ for the(each) center c of T. Then, first of all $a_c^{k-1}(u)$ is either isolated or does not exist, and secondly $a_r^k(u)$ exists for each $r \in R_u$ and further, since $D_r[a_r^k(u)]$ contains c, it also contains an endvertex x of a maximum length path. We therefore have $\partial_T(x,c) \geq \lfloor d/2 \rfloor > k-1 \geq \partial_T(u,c)$, so by Lemma 2.7 u is not strongly simplicial in T^k. Hence, both statements of the corollary are wrong in this case.

If $\partial_T(u,c) \geq k$ for a center c, then $a_c^k(u)$ exists. By Claim 1.2 we have $c \in V(p(u,r))$ for each $r \in R_u$, and since $k \leq \lceil (d-1)/2 \rceil$ we have that $a_r^k(u)$ also exists and further

$$a_c^{k-1}(u) = a_r^{k-1}(u) \text{ and } D_c[a_c^{k-1}(u)] = D_r[a_r^{k-1}(u)], \tag{2}$$

when T is rooted at c on one hand and at r on the other. By (2) the first statement is equivalent to k-strong simplicity and the corollary easily follows. □

Remark: When determining a strong elimination ordering of the vertices of the graph $G = T^k$ it suffices to only know the graph G, since in [7] a polynomial time algorithm is given to obtain the k-th root T from G (provided that we know G is a k-th power of a tree $G = T^k$. In fact, Chang et. al. [5] claim they can in this case obtain the k-th root in linear time). However, for a general graph H it is NP-hard to compute the k-th root H of $G = H^k$ as shown in [12].

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