# Egyptian Integral Domains

## Outline:

I. Egytian ~~~~ fractions

II. Egyptian domains and results from GLO.

III. My generalizations in domains

IV. Locally Egyptian domains and f.g. algebras

V. Rings w/ zerodivisors.



## I. Egyptian Fractions:

Ancient Egyptians (e.g. 1400 BCE, 1850 BCE) thought of fractions as sums of reciprocals of natural numbers.

e.g. $\frac{7}{13} = \frac{1}{2} + \frac{1}{26}$

Fibonacci (1203) showed that any pos. rational number can be rep'd in this way.

## II. Results & ideas from GLO [Guerrieri, Loper & Oman]

Def: [GLO]: An integral domain $D$, w/ fraction field $K$, is Egyptian if for any $0 \neq \alpha \in K$, $\exists$ distinct $d_1, \ldots, d_n \in D \setminus \{0\}$ such that

$$\alpha = \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_n}.$$

**Thm (essentially Fibonacci):**

$\mathbb{Z}$ is Egyptian.

**Thm 0 [GLO]:**

You don't have to say "distinct" in the def'n.

Hence, any domain generated by Egyptian elements is Egyptian.

**Observations:**

- Fields are Egyptian.
- If $D \subset E \subset K$ and $K = \mathrm{Frac}(D)$ and $D$ is Egyptian, so is $E$.

**Thms [GLO]** $A$ int. domain

$D$ is Egyptian if

1) $F[X] \subsetneq D \subseteq F(X)$, $F$ any field ← will generalize

2) $D$ is algebraic over an Egyptian subring. ← will partially generalize

3) $D = A[G]$, $G$ a tors-free abelian group, $A$ Egyp. domain ← will get a counterpart

4) Jacobson radical of $D$ is nonzero (e.g. any local or semilocal domain, or $D = A[[X]]$, $A$ any domain) ← generalizes to non-domains in II

**Thm 6 [GLO]:** For any int. domain $A$, $A[x]$ is not Egyptian. ← will generalize

## III. My generalizations of GLO.

**Thm 6** (generalizes GLO 1)
Let $D$ be Egyptian, $W \subseteq D[X]$ be a mult. closed set, and suppose $W \not\subseteq D$. Then $R = W^{-1}D[X]$ is Egyptian.

**Pf idea**: By GLO 0, need only check generators. So need only see that $X$ is Egyptian in $R$. Choose $g \in W$ of pos. degree.

If $X \mid g$, say $g = Xh$, then $X = \frac{g}{h} = \frac{1}{h/g} \in R$ ✓

Otherwise, $g = d + Xh$, $d \in D$ and $X = \frac{g}{h} - \frac{d}{h}$
$= \frac{1}{h/g} - \frac{1}{h}$ (Egyp. rep. of $d$) ✓

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Next we generalize GLO 5

**Prop**: Let $(G,+)$ be a tot. ordered abelian group, $\Gamma = \{\text{nonneg elts. of } G\}$. Let $D$ be a nontrivially $\Gamma$-graded integral domain. Then $D$ is not Egyptian.

**Lemma**: Let $\Lambda$ be an additive submonoid of $\mathbb{Q}^n$. Suppose $\mathbb{R}_{\geq 0}\Lambda$ is a group. Then $\Lambda$ is a group.

**Thm 7**: Let $\Lambda$ be an additive submonoid of $\mathbb{Q}^n$ that is not a group. $A$ int. dom., $D = A[\Lambda]$. Then $D$ has a nontrivial $\mathbb{R}_{\geq 0}$-grading.

— hence, by Prop, $D$ is not Egyptian

Next, look for a partial converse to GLO 2

**Lemma**: $R$ Egyp. domain, $I$ nonzero ideal. Then $\forall x \in R$, $\exists i_1, \ldots, i_n \in I$ (not nec. $\neq$) s.t. $x = \frac{1}{i_1} + \cdots + \frac{1}{i_n}$.

**Pf**: Let $0 \neq i \in I$

Then $ix = \frac{1}{d_1} + \cdots + \frac{1}{d_n}$

So $x = \frac{1}{id_1} + \cdots + \frac{1}{id_n}$.

**Thm 8** Let $R, S$ be int. domains w/ $R \subseteq S$ ~~and~~ that share a nonzero ideal (e.g. if $S$ is f.g. as an $R$-module). Then $R$ is Egyptian $\Leftrightarrow$ $S$ is Egyptian.

IV

# Locally Egyptian domains

**Def**: int dom. $R$ is *locally Egyptian* if $\exists$ $0 \neq a_1, \dots, a_n \in R$ such that $R[1/a_j]$ is Egyptian $\forall j$, and $(a_1, \dots, a_n) = R$.

**Thm 9**: Let $D$ be (locally) Egyptian, $K = \operatorname{Frac} D$, $L/K$ ext. ~~d~~field, $R$ f.g. $D$-algebra s.t. $D \subseteq R \subseteq L$. Then $R$ is locally Egyptian

**Cor**: Let $A$ be a field or $\mathbb{Z}$. Then any f.g. $A$-algebra domain is Egyptian.