Unit-Additive Rings

- joint w/ Jay Shapiro

Convention: All rings are commutative & unital

Notation: R : ring
$U(R)$ = units of R
$\mathfrak{N}(R)$ = nilradical of R
= nilpotent elements of R

Def/Prop: Let R be a ring. TFAE, in which case we say R is unit-additive (u-a.)

1) $\forall u, v \in U(R)$, either $u+v$ is a unit or is nilpotent
2) $\forall u \in U(R)$, $u+1 \in U(R) \cup \mathfrak{N}(R)$.
3) $U(R) \cup \mathfrak{N}(R)$ is a subring of R.
4) $U(R) \cup \mathfrak{N}(R)$ is a zero-dimensional, local subring of R
(So, if R is reduced (= nonzero), $U(R) \cup \{0\}$ is a field, called the field of units of R)
5) For any finite set $u_1, \ldots, u_k \in U(R)$, $\sum_{i=1}^{k} u_i$ is a unit or nilpotent

Ex: $k[x]$ is u-a., but $\mathbb{Z}$ is not.

Theorem (me): Any Euclidean domain that is not u-a. is Egyptian

Note: We have to allow for nilpotents, because if n is nilpotent, then 1 and $n-1$ are units

Outline:

I. Basics

II. U-additivity in group & semigroup algebras

III. u-additivity in affine algebras (connections w/ Alg. Geometry & (sort of) Liouville's Thm)

IV. Unit-additivity dimension (udim)

V. Rings of udim 1.

VI. The u-additive closure

I. Basics:

- R is u-a. $\Leftrightarrow$ $R[x]$ is u-a
- $R[x, x^{-1}]$ is never u-a. (e.g. $1+x$ can't be a unit or nilpotent)
- Any u-a. ring R has Jac. radical of $R = \mathfrak{N}(R)$.

Prop: If R is a positively graded ring, R is u-a. $\Leftrightarrow$ R_0 is u.a.

Prop: S u-a., k = field of units, R k-subalgebra of S $\Rightarrow$ R u-a.

Prop: $R \to S$ integral extension of rings, S u-a. $\Rightarrow$ R u-a

(Converse is false)

Prop: $R = S \times T$ is u-a.

$\iff$

$U(S) = \{1_S\}$ and $U(T) = \{1_T\}$.

Prop: R ring, $I \subseteq \mathfrak{N}(R)$ ideal

R u-a. $\iff$ R/I u-a.

II. Group & Monoid algebras:

Prop: Let M be a commutative, cancellative, torsion-free monoid. (e.g. $M \leq \mathbb{Z}^n$), A any ring

$A[M]$ is u-a. $\iff$ A is u-a. & M is positive.

Cor: When G is a torsion-free group,

$A[G]$ is never u-a unless G trivial

Prop: Let M be a comm. cancellative monoid, A a ring of prime char $p > 0$

Let M_p = the p-torsion subgroup of M.

Then $A[M]$ is u-a. $\iff$ $A[M/M_p]$ is u-a.

III. Affine algebras

Prop: $R = S/P$, $P \in \operatorname{Spec} S$, S any ring.

R is u.a. $\iff P \cup \widehat{1+P}$ is a ring.

Thm: Let R, S, P as above but $S = k[x_1, \ldots, x_n]$, k alg closed field.

R is u-a. $\iff k + P = P \cup \widehat{1+P}$.

Theorem (Alg. geom. characterization of unit additivity)

Let R be an int. domain that is a f.g. algebra over an alg. closed field k (i.e. coord ring of an affine variety V over k)

Let $A = k[t, t^{-1}]$ (i.e. coord ring of $G_m = (\mathbb{A}^1_k) \setminus \{0\}$)

Exactly one of the following holds

1) R is u-a., $U(R) = k^\times$, and the only k-algebra maps $A \to R$ send $t \mapsto \alpha$ for some $\alpha \in k^\times$. None of these are injective. (corresp. to const maps $V \to G_m$)

2) R is not u-a., and there is an injective k-alg. map $A \xrightarrow{\varphi} R$. For any such φ the corresp map $V \to G_m$ has cofinite image.

IV: Unit-additivity dimension:

Let R be a not u-a. ring.

Assume domain.

Let W = nonzero sums of units in R.

One might hope $W^{-1}R$ is u-a.

(e.g. if $R=\mathbb{Z}$, then $W=\mathbb{Z}\setminus\{0\}$, $W^{-1}R=\mathbb{Q}$, u-a.)

But it's not true.

Ex: $\mathbb{Z}[x, \frac{2}{x}]$ goes to $\mathbb{Q}[x,x^{-1}]$, which is not u-a.

Iterative construction:

R rings with $\mathfrak{N}(R)\in \operatorname{Spec} R$.

$W_0=\{1\}$

$V_0=\widetilde{W_0}=U(R)$

For each $i\geq 1$, W_i = non-nilp sums of elts in V_{i-1}

$V_i=\widetilde{W_i}$.

We say $\underline{\operatorname{udim} R\leq n}$ if $W_n^{-1}R$ is u-a.

Thm: $\left(\bigcup_{n\geq 0} W_n\right)^{-1}R$ is always u-a.

We have examples of f.g. k-algebras of all finite udims

We have a k algebra of infinite udim.

Thm:
R int. domain
f.g. over a field, then
$\mathrm{udim} R \leq \dim R$.

V. Rings of udim 1: Assume domain

- $\mathrm{Jac}(R) \neq 0 \implies \mathrm{udim} R = 1$.
- R semilocal of pos dim $\implies \mathrm{udim} = 1$
- $R = A[[X]] \implies \mathrm{udim} = 1$ (A any ~~ring~~ domain)
- Any subring of the ring of algebraic integers has $\mathrm{udim} = 1$.

VI: Unit-additive closure

R domain

$W = \bigcup_{n \geq 0} W_n$

~~WE~~ $\forall$ injective ring homs $R \xrightarrow{\varphi} S$

with S u-a domain,

$\exists ! \tilde{\varphi} : W^{-1}R \to S$ extending

φ along the localization map, also injective.