Permutation statistics and moment sequences

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Abstract

Which combinatorial sequences correspond to moments of probability measures on the real line? We present a generating function $\mathcal{C}$, as a continued fraction, for a 14-parameter family of integer sequences and interpret these in terms of statistics on permutations and other combinatorial objects. Special cases include several classical and noncommutative probability laws, and a substantial subset of the orthogonalizing measures in the q-Askey scheme of orthogonal polynomials.

Under mild conditions on the parameters, the sequences arising from $\mathcal{C}$ are moment sequences, and this continued fraction captures a variety of combinatorial sequences. In particular, it characterizes the moment sequences associated to the numbers of permutations avoiding patterns of length three. A permutation of the integers $1, 2, \ldots, n$ avoids a pattern P (which is also a permutation) if it contains no subsequence whose integers appear in the same order of size as those in P. Avoiding 123, for example, means not having any three entries in increasing order.

An intriguing question, currently occupying several teams of researchers, is the potential significance of combinatorial sequences being moment sequences, in particular because this strengthens aspects previously studied by combinatorialists, such as log-convexity, which can be used to obtain bounds on growth rates.

The fourteen combinatorial statistics further generalize to colored permutations, where each entry has a color associated to it, and, as an infinite family of statistics, to the k-arrangements: permutations with k-colored fixed points, introduced here. This is joint work with Natasha Blitvić, Lancaster University.

Keywords: Continued fraction, orthogonal polynomials, permutation avoiding pattern, log-convexity, colored permutation.