Groups, Lattices, and Closure operators

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Abstract

A *lattice* is a partially ordered set in which every pair of elements has a least upper bound and a greatest lower bound. The collection of all subgroups of a group forms a lattice under inclusion. In fact, every lattice can be realized as a sublattice of the lattice of all subgroups of a group, though not every lattice is a full subgroup lattice.

Given a partially ordered set P, we can define a closure operator cl on the set. This is a function from P to itself such that $x \leq \mathrm{cl}(x)$ for all x, if $x \leq y$ then $\mathrm{cl}(x) \leq \mathrm{cl}(y)$, and $\mathrm{cl}(\mathrm{cl}(x)) = \mathrm{cl}(x)$. The collection of all possible closure operators on a lattice is itself a lattice.

Suppose we start with the lattice L of all subgroups of a finite group G. We can then construct the lattice of all closure operators on L. When, if ever, will this be the lattice of all subgroups of some group K?

In joint work with Martha Kilpack, we answered this question, as well as the related one when G is infinite (there is a subtle issue that needs to be addressed then).

Keywords: partially ordered set, closure operator, lattice of subgroups.