The Fundamental Theorem of Calculus Part II.

A. Functions of Bounded Variation.

Definition 0.1 Given f on $[a, b]$, a closed, finite interval, and $P = \{x_0, x_1, \ldots, x_k\}$ a partition of $[a, b]$, we define the *variation of f with respect to P* by

$$V(f, P) = \sum_{k=1}^{n} |f(x_i) - f(x_{i-1})|$$

and the *total variation* of f on $[a, b]$ by

$$TV(f) = \sup\{V(f, P)\colon\ P \text{ a partition of } [a, b]\}.$$

A real-valued function f defined on $[a, b]$ has *bounded variation* on $[a, b]$ if $T(f) < \infty$.

Examples. (1) If f is monotone on $[a, b]$ then $TV(f) = |f(b) - f(a)| < \infty$, so that f is BV on $[a, b]$. This implies that the Cantor function $\varphi(x)$ is BV on $[0, 1]$.
(2) If f is Lipschitz on $[a, b]$ then $TV(f) \leq c(b - a)$ where c is the Lipschitz constant of f.
(3) $f(x) = \sin(1/x)$ on $(0, 1]$ with $f(0) = 0$ is not BV on $[0, 1]$.
(4) $f(x) = x \sin(1/x)$ is continuous on $[0, 1]$ but not BV on $[0, 1]$.

Definition 0.2 Given f BV on $[a, b]$, we define the *total variation function* of f by $TV(f|_{[a,x]})$. Letting $TV(f|_{[a,a]}) = 0$ and $TV(f|_{[a,b]}) = TV(f)$ we see that the total variation function is defined for all $x \in [a, b]$.

Lemma 0.1 If f is BV on $[a, b]$ then the total variation function of f is real-valued and increasing on $[a, b]$ and moreover, the function $f(x) + TV(f|_{[a,x]})$ is real-valued and increasing on $[a, b]$.

Theorem 0.1 (Jordan) A function f is BV on $[a, b]$ if and only if it can be written as the difference of two increasing functions on $[a, b]$.

B. Absolutely Continuous Functions.

Definition 0.3 A real-valued function f on $[a,b]$ is *absolutely continuous* on $[a,b]$ provided that for every $\epsilon > 0$ there is a $\delta > 0$ such that if $\{(a_k, b_k)\}_{k=1}^n$ is a finite, disjoint collection of open intervals in (a,b) then

$$\sum_{k=1}^{n}(b_k - a_k) < \delta \Longrightarrow \sum_{k=1}^{n}|f(b_k) - f(a_k)| < \epsilon.$$

Remark 0.1 (1) If f is AC on $[a,b]$ then f is continuous on $[a,b]$. However, the converse is false.

(2) **Claim:** If f is AC on $[a,b]$, then f is BV on $[a,b]$.

(3) Since the function $f(x) = x\ \sin(1/x)$ is continuous on $[0,1]$ but not BV on $[0,1]$, it is not AC on $[0,1]$.

(4) The Cantor function $\varphi(x)$ is not AC on $[0,1]$. We will prove this in detail but the reason for this is the following. $\varphi(x)$ increases on $[0,1]$ from $\varphi(0) = 0$ to $\varphi(1) = 1$, but its derivative vanishes off the Cantor set C which has measure zero. This means that φ has to do all of its "climbing" on a set of measure zero, and its total variation over any finite collection of intervals containing C must be 1. This example means that there are monotone, continuous functions that are not AC.

(5) If f is Lipschitz on $[a,b]$ then f is AC on $[a,b]$. However, the converse is false as can be seen by considering the function $f(x) = \sqrt{x}$ on $[0,1]$. This function is not Lipschitz but is AC on $[0,1]$.

Theorem 0.2 If f is AC on $[a,b]$ then f can be written as the difference of increasing functions, both absolutely continuous on $[a,b]$.

The key to proving this Theorem is the following claim.

Claim: If f is AC on $[a,b]$ then so is its total variation function.

C. The Fundamental Theorem Part II.

Theorem 0.3 If f is AC on $[a,b]$ then f is differentiable a.e. on $[a,b]$ and its derivative satisfies

$$\int_a^b f' = f(b) - f(a).$$

The proof of this Theorem relies on the following Lemma.

Lemma 0.2 Suppose that f is continuous on $[a,b]$, and for each $h > 0$ define the divided difference function $g_h(x)$ on $[a,b]$ by

$$g_h(x) = \frac{f(x+h) - f(x)}{h}$$

(where we assume that f has been extended to $[a,\infty)$ by letting $f(x) = f(b)$ for $x > b$). Then f is absolutely continuous if and only if the collection $\{g_h\}_{0<h\leq 1}$ is uniformly integrable over $[a,b]$.

In fact we can extend the previous Theorem as follows.

Theorem 0.4 A function f defined on a closed, bounded interval $[a,b]$ is AC on $[a,b]$ if and only if f is an indefinite integral over $[a,b]$, that is, if and only if f can be written

$$f(x) = f(a) + \int_a^x g$$

for some function g integrable over $[a,b]$.

Remark 0.2 (1) If f is BV on $[a,b]$ then it follows from the Lebesgue Differentiation Theorem and Jordan's Theorem that f is differentiable a.e. and that f' is integrable on $[a,b]$.
(2) We would like to say that $f(x) = \int_a^x f' + f(a)$ for all $x \in [a,b]$, but we know that this is not necessarily the case. However the previous theorem implies that the function $g(x) = \int_a^x f'$ is absolutely continuous.
(3) What can we say about the remainder $h(x) = f(x) - g(x) = f(x) - \int_a^x f'$? Since f' is integrable on $[a,b]$, the FTC Part I tells us that $h'(x) = 0$ a.e. Such a function is called *singular* on $[a,b]$
(4) We conclude that if f is BV on $[a,b]$ then it can be written as $f = g + h$ where g is absolutely continuous on $[a,b]$ and h is singular on $[a,b]$. This is known as the *Lebesgue decomposition* of f.