Homework for Lecture 9. Due 11-12-07.

Exercise 1. Prove parts (a)-(c) in the theorem below.

Theorem 1 *Let $I = [a, a+l]$. Then the following collections are orthonormal bases for $L^2(I)$.*

(a) $\left\{\sqrt{\frac{2}{l}}\cos\frac{2k+1}{2l}\pi(x-a)\right\}_{k=0}^{\infty}$.

(b) $\left\{\sqrt{\frac{2}{l}}\sin\frac{2k+1}{2l}\pi(x-a)\right\}_{k=0}^{\infty}$.

(c) $\left\{\sqrt{\frac{2}{l}}\sin\frac{k}{l}\pi(x-a)\right\}_{k=1}^{\infty}$.

(d) $\left\{\sqrt{\frac{2}{l}}\cos\frac{k}{l}\pi(x-a)\right\}_{k=0}^{\infty}$.

Exercise 2. Prove that the operator P_I defined below is self-adjoint, that is, that $P_I^* = P_I$. This means that for every f, $g \in L^2(\mathbf{R})$, $\langle P_I f, g\rangle = \langle f, P_I g\rangle$.

Definition 1 *Given I, ϵ, ϵ' as above, define the operator P_I on $L^2(\mathbf{R})$ by*

$$P_I f(x) = b_I(x)[b_I(x)f(x) \pm b_I(2\alpha - x)f(2\alpha - x) \pm b_I(2\beta - x)f(2\beta - x)].$$

Exercise 3. Prove that the system $\{b_I\, e_k\}$ as defined in the theorem below is an orthonormal system in $L^2(\mathbf{R})$.

Theorem 2 *If I is given as above and if b_I is a smooth bell function over I, then the collection $\{b_I(x)\, e_k(x)\}_k$ is an orthonormal basis for the subspace $P_I(L^2(\mathbf{R}))$.*

Exercise 4. Prove the following Lemma.

Lemma 1 *Suppose that $g \in L^2(\mathbf{R})$ is even or odd about α and β in some combination, and that the smooth projector P_I has the same polarity at α and β. Then $P_I(b_I\, g) = b_I\, g$.*

Exercise 5. Prove that the systems $\{\theta_{j,k}\}$ defined in the theorem below are in fact orthonormal systems. Really the only thing necessary to prove is the orthogonality of functions in the systems that correspond to adjacent intervals.

Theorem 3 *Choose a sequence $\{\alpha_j\}_{j\in\mathbf{Z}}$ strictly increasing and going to infinity in both directions and numbers ϵ_j satisfying $\alpha_j + \epsilon_j \le \alpha_{j+1} - \epsilon_{j+1}$. Now choose smooth projections P_j onto $I_j = [\alpha_j, \alpha_{j+1}]$ with opposite polarities at the common endpoints. Let $\theta_{j,k}(x) = b_{I_j}(x)\, e_{j,k}(x)$ where $\{e_{j,k}(x)\}_k$ is the sine or cosine basis on $L^2(I_j)$ with the same polarity as P_j at each endpoint. Then $\{\theta_{j,k}\}$ is an orthonormal basis for $L^2(\mathbf{R})$.*