The Daubechies Wavelets.

A. Vanishing Moments.

Related to smoothness:

Theorem 0.1 *Let $\psi(x)$ be such that for some $N \in \mathbf{N}$, both $x^N \psi(x)$ and $\gamma^{N+1} \widehat{\psi}(\gamma)$ are in $L^1(\mathbf{R})$. If $\{\psi_{j,k}(x)\}_{j,k \in \mathbf{Z}}$ is an orthogonal system on $\mathbf{R}$, then $\int_{\mathbf{R}} x^m \, \psi(x) \, dx = 0$ for $0 \leq m \leq N$.*

Related to approximation of smooth functions:

Theorem 0.2 *Given $N \in \mathbf{N}$, assume that the function $f \in C^N(\mathbf{R})$, and that $f^{(N)} \in L^\infty(\mathbf{R})$. Assume that the function $\psi(x)$ has compact support, that $\int_{\mathbf{R}} x^m \, \psi(x) \, dx = 0$, for $0 \leq m \leq N-1$ and that $\int_{\mathbf{R}} |\psi_{j,k}(x)|^2 \, dx = 1$ for all j, $k \in \mathbf{Z}$. Then there is a constant $C > 0$ depending only on N and $f(x)$ such that for every j, $k \in \mathbf{Z}$, $|\langle f, \psi_{j,k} \rangle| \leq C 2^{-jN} \, 2^{-j/2}$.*

Related to reproduction of polynomials:

Theorem 0.3 *Let $\varphi(x)$ be a compactly supported scaling function associated with an MRA, and let $\psi(x)$ be the wavelet. If $\psi(x)$ has N vanishing moments, then for each integer $0 \leq k \leq N-1$, there are coefficients $\{q_{k,n}\}_{n \in \mathbf{Z}}$ such that $\sum_n q_{k,n} \, \varphi(x+n) = x^k$.*

Equivalent conditions for vanishing moments:

Theorem 0.4 *Let $\varphi(x)$ be a compactly supported scaling function associated with an MRA with finite scaling filter $h(n)$. Let $\psi(x)$ be the corresponding wavelet. Then for each $N \in \mathbf{N}$, the following are equivalent.*

(a) $\displaystyle\int_{\mathbf{R}} x^k \, \psi(x) \, dx = 0$ *for* $0 \leq k \leq N-1$.

(b) $m_0^{(k)}(1/2) = 0$, *for* $0 \leq k \leq N-1$.

(c) $m_0(\gamma)$ *can be factored as* $m_0(\gamma) = \left(\dfrac{1+e^{-2\pi i\gamma}}{2}\right)^N \mathcal{L}(\gamma)$ *for some period* 1 *trigonometric polynomial* $\mathcal{L}(\gamma)$.

(d) $\displaystyle\sum_n h(n) \, (-1)^n \, n^k = 0$ *for* $0 \leq k \leq N-1$.

B. Daubechies Polynomials.

(1) We want to construct a trig polynomial $m_0(\gamma) = \frac{1}{\sqrt{2}} \sum_k h(k) \, e^{-2\pi i k\gamma}$ satisfying $m_0(\gamma) = \left(\frac{1+e^{-2\pi i\gamma}}{2}\right)^N \mathcal{L}(\gamma)$. and satisfying the QMF conditions.

(2) $|m_0(\gamma)|^2 = \left|\dfrac{1+e^{-2\pi i\gamma}}{2}\right|^{2N} |\mathcal{L}(\gamma)|^2 = \cos^{2N}(\pi\gamma)\, L(\gamma)$.

(3) Since $L(\gamma)$ is a real-valued trig polynomial with real coefficients, we arrive at $L(\gamma) = P(\sin^2(\pi\gamma))$ for some polynomial P.

(4) This polynomial P must satisfy $1 = (1-y)^N P(y) + y^N P(1-y)$ with $P(y) \geq 0$ for all $0 \leq y \leq 1$.

(5) We arrive at finally the definition

$$P_{N-1}(y) = \sum_{k=0}^{N-1} \binom{2N-1}{k} y^k (1-y)^{N-1-k}.$$

For example,

$$\begin{aligned} P_0(y) &= 1, \\ P_1(y) &= 1+2y, \\ P_2(y) &= 1+3y+6y^2, \\ P_3(y) &= 1+4y+10y^2+20y^3. \end{aligned}$$

C. Some simple examples.

If $m_0(\gamma) = \left(\frac{1+e^{-2\pi i\gamma}}{2}\right)^N \mathcal{L}(\gamma)$ then $|m_0(\gamma)|^2 = \cos^{2N}(\pi\gamma)\, P_{N-1}(\sin^2 \pi\gamma)$, so if we solve $|\mathcal{L}(\gamma)|^2 = P_{N-1}(\sin^2 \pi\gamma)$ then we have $m_0(\gamma)$ and hence our scaling sequence $\{h(n)\}$.

(a) $N = 1$. $P_0(y) = 1$ so $\mathcal{L}(\gamma) = 1$ and

$$m_0(\gamma) = \left(\frac{1+e^{-2\pi i\gamma}}{2}\right) = \frac{1}{2} + \frac{1}{2}\, e^{-2\pi i\gamma} = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\, e^{-2\pi i\gamma}\right)$$

so $h(0) = h(1) = 1/\sqrt{2}$ and $h(n) = 0$ otherwise and we have recovered the Haar system.

(b) $N = 2$. Here $P_1(y) = 1+2y$ so that we must solve

$$|\mathcal{L}(\gamma)|^2 = 1 + 2\, \sin^2 \pi\gamma = 1 + 2((1-\cos(2\pi\gamma))/2) = 2 - \cos(2\pi\gamma).$$

If $\mathcal{L}(\gamma) = a + b\, e^{-2\pi i\gamma}$ then

$$|\mathcal{L}(\gamma)|^2 = (a^2+b^2) + 2ab\, \cos(2\pi\gamma).$$

This, together with the normalization $m_0(0) = \mathcal{L}(0) = a + b = 1$ leads to the nonlinear system

$$\begin{aligned} a^2+b^2 &= 2, \\ 2ab &= -1, \\ a+b &= 1 \end{aligned}$$

which when solved gives the Daubechies 4–coefficient wavelet filter.

D. Spectral Factorization.

1. Look directly at the equation $|m_0(\gamma)|^2 = \cos^{2N}(\pi\gamma)\, P_{N-1}(\sin^2(\pi\gamma))$.

2. Extend to the complex plane by making the substitution $z = e^{2\pi i\gamma}$. Then

$$\cos(2\pi\gamma) = \frac{e^{2\pi i\gamma} + e^{-2\pi i\gamma}}{2} = \frac{z + z^{-1}}{2},$$

$$\sin^2(\pi\gamma) = \frac{1}{2}\,(1 - \cos(2\pi\gamma)) = \frac{1}{2} - \frac{z + z^{-1}}{4},$$

$$\cos^2(\pi\gamma) = \frac{1}{2}\,(1 + \cos(2\pi\gamma)) = \frac{1}{2} + \frac{z + z^{-1}}{4}.$$

3. Now we can define

$$\begin{aligned} \cos^{2N}(\pi\gamma)\, P_{N-1}(\sin^2(\pi\gamma)) &= \left(\frac{1}{2} + \frac{z + z^{-1}}{4}\right)^N P_{N-1}\left(\frac{1}{2} - \frac{z + z^{-1}}{4}\right) \\ &= \sum_{m=-2N+1}^{2N-1} a_m\, z^m \equiv \mathbf{P}_{2N-1}(z). \end{aligned}$$

4. So that we are dealing with a true polynomial, we can now define

$$\widetilde{\mathbf{P}}_{4N-2}(z) = z^{2N-1}\, \mathbf{P}_{2N-1}(z) = \sum_{m=0}^{4N-2} a_{m+2N-1}\, z^m$$

and deal directly with the following factorization problem: *Find a polynomial* $B_{2N-1}(z)$ *satisfying* $|B_{2N-1}(z)|^2 = \widetilde{\mathbf{P}}_{4N-2}(z)$. In this case we would have $m_0(\gamma) = B_{2N-1}(e^{2\pi i\gamma})$.

Theorem 0.5 $\widetilde{\mathbf{P}}_{4N-2}(z) = |B_{2N-1}(z)|^2$ *where*

$$B_{2N-1}(z) = const.\,(z+1)^N \prod_{z_0 \in Z_{\mathbf{R}}} (z - z_0) \prod_{z_0 \in Z_{\mathbf{C}}} (z - z_0)(z - \overline{z_0})$$

where

$$Z_{\mathbf{R}} = \{z_0 \in \mathbf{R} \colon \widetilde{\mathbf{P}}_{4N-2}(z_0) = 0,\ |z_0| < 1\}$$

and

$$Z_{\mathbf{C}} = \{z_0 \in \mathbf{C} \colon \widetilde{\mathbf{P}}_{4N-2}(z_0) = 0,\ |z_0| < 1,\ \Im(z_0) > 0\}.$$

E. Further Examples.

(a) $N = 2$.

$$\widetilde{\mathbf{P}}_6(z) = \frac{1}{32}\,(-1 + 9z^2 + 16z^3 + 9z^4 - z^6).$$

We factor

$$\widetilde{\mathbf{P}}_6(z) = \frac{1}{32}\,(z+1)^4\,(-z^2 + 4z - 1) - \frac{1}{32}\,(z+1)^4\,(z - (2 - \sqrt{3}))\,(z - (2 + \sqrt{3})).$$

Therefore,

$$\begin{aligned} B_3(z) &= \frac{1}{4\sqrt{2}}\,(z+1)^2\,(2-\sqrt{3})^{-1/2}\,(z-(2-\sqrt{3})) \\ &= \frac{1+\sqrt{3}}{8}\,(z+1)^2\,(z-(2-\sqrt{3})) \\ &= \frac{1+\sqrt{3}}{8}\,z^3 + \frac{3+\sqrt{3}}{8}\,z^2 + \frac{3-\sqrt{3}}{8}\,z + \frac{1-\sqrt{3}}{8}. \end{aligned}$$

(b) $N = 3$.

$$\widetilde{\mathbf{P}}_{10}(z) = \frac{1}{512}(3 - 25\,z^2 + 75\,z^4 + 256\,z^5 + 75\,z^6 - 25\,z^8 + 3\,z^{10}).$$

We factor

$$\begin{aligned} \widetilde{\mathbf{P}}_{10}(z) &= \frac{1}{512}\,(z+1)^6(3z^4 - 18z^3 + 38z^2 - 18z + 3) \\ &= \frac{3}{512}\,(z+1)^6\,(z-\alpha)\,(z-\overline{\alpha})(z-\alpha^{-1})\,(z-\overline{\alpha}^{-1}), \end{aligned}$$

where $\alpha \approx .2873 + .1529\,i$ and

$$B_5(z) = \frac{\sqrt{3}}{|\alpha| 16\sqrt{2}}\,(z+1)^3\,(z-\alpha)\,(z-\overline{\alpha})$$