Wavelet Orthonormal Bases for $L^2(\mathbf{R})$.

A. Wavelet systems.

Definition 0.1 *A* wavelet system *in $L^2(\mathbf{R})$ is a collection of functions of the form*

$$\{D_{2^j}T_k\psi\}_{j,k\in\mathbf{Z}} = \{2^{j/2}\psi(2^jx-k)\}_{j,k\in\mathbf{Z}} = \{\psi_{j,k}\}_{j,k\in\mathbf{Z}}$$

where $\psi \in L^2(\mathbf{R})$ is a fixed function sometimes called the mother wavelet.

A wavelet system that forms an orthonormal basis for $L^2(\mathbf{R})$ is called a wavelet orthonormal basis *for $L^2(\mathbf{R})$.*

Remarks. (a) If the mother wavelet $\psi(x)$ is "concentrated" around 0 then $\psi_{j,k}(x)$ is concentrated around $2^{-j}k$. If $\psi(x)$ is essentially supported on an interval of length L, then $\psi_{j,k}(x)$ is essentially supported on an interval of length $2^{-j}L$. In fact, if $\psi(x)$ is concentrated on an interval I, then $\psi_{j,k}(x)$ is concentrated on the interval $2^{-j}I + 2^{-j}k$.

(b) Since $(D_{2^j}T_k\psi)^\wedge(\gamma) = D_{2^{-j}}M_k\widehat{\psi}(\gamma)$ it follows that if $\widehat{\psi}$ is concentrated on the interval I then $\widehat{\psi_{j,k}}$ is concentrated on the interval 2^jI.

(c) A wavelet basis then corresponds to a dyadic tiling of the time-frequency plane.

B. Example: The Haar system.

Definition 0.2 *For each pair of integers j, $k \in \mathbf{Z}$, define the interval $I_{j,k}$ by*

$$I_{j,k} = [2^{-j}k, 2^{-j}(k+1)).$$

The collection of all such intervals is called the collection of dyadic subintervals of $\mathbf{R}$. *We write $I_{j,k} = I^\ell_{j,k} \cup I^r_{j,k}$, where $I^\ell_{j,k}$ and $I^r_{j,k}$ are dyadic intervals at scale $j+1$, to denote the left half and right half of the interval $I_{j,k}$. In fact, $I^\ell_{j,k} = I_{j+1,2k}$ and $I^r_{j,k} = I_{j+1,2k+1}$.*

Lemma 0.1 *Given j_0, k_0, j_1, $k_1 \in \mathbf{Z}$, with either $j_0 \neq j_1$ or $k_0 \neq k_1$, then either I_{j_1,k_1} and I_{j_0,k_0} are disjoint or one is contained in the other. In the latter case, the smaller interval is contained in either the right half or left half of the larger.*

Definition 0.3 *A* dyadic step function *is a step function $f(x)$ with the property that for some $j \in \mathbf{Z}$, $f(x)$ is constant on all dyadic intervals $I_{j,k}$, $k \in \mathbf{Z}$. We say in this case that $f(x)$ is a scale j dyadic step function.*

Remarks. (a) For each $j \in \mathbf{Z}$, the collection of all scale j dyadic step functions is a linear space.

(b) If $f(x)$ is a scale j dyadic step function on an interval I, then $f(x)$ is also a scale j' dyadic step function on I for any $j' \geq j$.

Definition 0.4 *Let* $p(x) = \chi_{[0,1)}(x)$. *For each* $j,\, k \in \mathbf{Z}$, *define*

$$p_{j,k}(x) = 2^{j/2}\, p(2^j x - k) = D_{2^j} T_k p(x).$$

The collection $\{p_{j,k}(x)\}_{j,k\in\mathbf{Z}}$ *is referred to as the system of* Haar scaling functions. *For each* $j \in \mathbf{Z}$, *the collection* $\{p_{j,k}(x)\}_{k\in\mathbf{Z}}$ *is referred to as the system of* scale j Haar scaling functions.

Let $h(x) = \chi_{[0,1/2)}(x) - \chi_{[1/2,1)}(x)$, *and for each* $j,\, k \in \mathbf{Z}$, *define*

$$h_{j,k}(x) = 2^{j/2}\, h(2^j x - k) = D_{2^j} T_k h(x).$$

The collection $\{h_{j,k}(x)\}_{j,k\in\mathbf{Z}}$ *is referred to as the* Haar system on $\mathbf{R}$. *For each* $j \in \mathbf{Z}$, *the collection* $\{h_{j,k}(x)\}_{k\in\mathbf{Z}}$ *is referred to as the system of* scale j Haar functions.

Remark. (a) For each $j,\, k \in \mathbf{Z}$,

$$p_{j,k}(x) = 2^{j/2}\, \chi_{I_{j,k}}(x),$$

$$h_{j,k}(x) = 2^{j/2}\, (\chi_{I^{\ell}_{j,k}}(x) - \chi_{I^{r}_{j,k}}(x)) = 2^{j/2}\, (\chi_{I_{j+1,2k}}(x) - \chi_{I_{j+1,2k+1}}(x)).$$

Both $p_{j,k}(x)$ and $h_{j,k}(x)$ are supported on the interval $I_{j,k}$ and neither one vanishes on that interval. We associate to each interval $I_{j,k}$ the pair of functions $p_{j,k}(x)$ and $h_{j,k}(x)$.

(b) For each $j,\, k \in \mathbf{Z}$, $p_{j,k}(x)$ is a scale j dyadic step function (hence also a scale $j+1$ dyadic step function), and $h_{j,k}(x)$ is a scale $j+1$ dyadic step function.

Theorem 0.1 *The Haar system is an orthonormal system on* $\mathbf{R}$ *and for each* $j \in \mathbf{Z}$, *the scale* j *Haar scaling functions, form an orthonormal system on* $\mathbf{R}$.

Lemma 0.2 *(The Splitting Lemma.) Let* $j \in \mathbf{Z}$, *and let* $g_j(x)$ *be a scale* j *dyadic step function. Then* $g_j(x)$ *can be written as* $g_j(x) = r_{j-1}(x) + g_{j-1}(x)$, *where* $r_{j-1}(x)$ *has the form*

$$r_{j-1}(x) = \sum_k a_{j-1}(k)\, h_{j-1,k}(x),$$

for some coefficients $\{a_{j-1}(k)\}_{k\in\mathbf{Z}}$, *and* $g_{j-1}(x)$ *is a scale* $j-1$ *dyadic step function.*

Moreover, $g_{j-1}(x)$ *and* $r_{j-1}(x)$ *are orthogonal.*

Theorem 0.2 *The Haar system is an orthonormal basis for* $L^2(\mathbf{R})$.

C. Characterization of orthonormal wavelet bases.

Lemma 0.3 *Suppose that a sequence* $\{x_n\}$ *in a Hilbert space* H *satisfies*

1. *for all* $x \in H$, $\sum_n |\langle x, x_n\rangle|^2 = \|x\|^2$, *and*
2. $\|x_n\| = 1$ *for all* n.

Then $\{x_n\}$ *is an orthonormal basis for* H.

Remarks. (a) Obviously the above theorem characterizes orthonormal bases. In other words the theorem is an "if and only if" theorem.

(b) Each of the conditions in the theorem is required. For example if we take the system $\{(1/\sqrt{2})e^{2\pi inx},\ (1/\sqrt{2})e^{2\pi i(n+1/2)x}\}_{n\in\mathbf{Z}}$, it is the union of two orthonormal bases and hence satisfies 1. However, it clearly does not satisfy 2. Such a system is called a *tight frame*. More about this later in the semester.

Theorem 0.3 *Suppose that $\psi \in L^2(\mathbf{R})$ satisfies $\|\psi\|_2 = 1$. Then $\{\psi_{j,k}\}_{j,k\in\mathbf{Z}}$ is an orthonormal wavelet basis for $L^2(\mathbf{R})$ if and only if*

1. $\sum_{j\in\mathbf{Z}} |\widehat{\psi}(2^j\gamma)|^2 \equiv 1$, *and*

2. $\sum_{j=0}^{\infty} \widehat{\psi}(2^j\gamma)\,\overline{\widehat{\psi}(2^j(\gamma+k))} = 0$ *for all odd integers k.*

Remark. Condition 1. above says that in order for a wavelet system to be an orthonormal basis, the dilated Fourier transforms of the mother wavelet must "cover" the frequency axis. So for example if $\widehat{\psi}$ had very small support, then it could never generate a wavelet orthonormal basis.

Theorem 0.4 *Given $\psi \in L^2(\mathbf{R})$, the wavelet system $\{\psi_{j,k}\}_{j,k\in\mathbf{Z}}$ is an orthonormal system in $L^2(\mathbf{R})$ if and only if*

1. $\sum_{k\in\mathbf{Z}} |\widehat{\psi}(\gamma+k)|^2 \equiv 1$, *and*

2. $\sum_{j\in\mathbf{Z}} \widehat{\psi}(2^j(\gamma+k))\,\overline{\widehat{\psi}(\gamma+k)} = 0$ *for all $j \geq 1$.*

Remark. Combining Condition 1. above with Condition 1. of the previous theorem implies that a function generating an orthoormal wavelet basis must "tile" the frequency axis both by dilation and by translation.

D. Example: The Bandlimited Wavelet.

Theorem 0.5 *Let $\psi \in L^2(\mathbf{R})$ be defined by $\widehat{\psi} = \chi_{[-1,-1/2]} + \chi_{[1/2,1]}$. Then $\{\psi_{j,k}\}$ is a wavelet orthonormal basis for $L^2(\mathbf{R})$.*

Remarks. (a) Here we can give a very clear interpretation of the meaning of the wavelet coefficients $\{\langle f, \psi_{j,k}\rangle\}_{k\in\mathbf{Z}}$ as the Fourier coefficients of $\widehat{f}$ cut-off to the interval $[-2^j, -2^{j-1}] \cup [2^{j-1}, 2^j]$, so that those coefficients all together capture the features of f at "scale" 2^{-j}. Coefficients at different values of k identify the intensity of that range of frequencies at "time" $2^{-j}k$.

(b) Returning to the idea of the wavelet basis as representing a tiling of the time-frequency plane, we can see that the elements of the Haar system have perfect time-localization but very poor frequency localization. The bandlimited wavelet system has perfect frequency localization and poor time localization. Can we find a wavelet orthonormal basis that does better?

E. Example: The Meyer wavelet.

The idea here is to find a mother wavelet similar to but smoother than the Bandlimited wavelet. The specific construction follows certain steps.

1. Define the functions $s(x)$ and $c(x)$ with the following properties:

   (a) $s(x) = 0$ and $c(x) = 1$ if $x \le -1/6$,

   (b) $s(x) = 1$ and $c(x) = 0$ if $x \ge 1/6$,

   (b) $0 \le s(x),\ c(x) \le 1$ for all x, and

   (b) $s(x)^2 + c(x)^2 = 1$ for all x.

2. Define $\psi(x)$ by means of its Fourier transform by

$$\widehat{\psi}(\gamma) = -e^{-\pi i \gamma} \begin{cases} 0 & \text{if } |\gamma| \le 1/3 \text{ or } |\gamma| \ge 4/3, \\ s(\gamma - 1/2) & \text{if } \gamma \in [1/3, 2/3], \\ c(\gamma/2 - 1/2) & \text{if } \gamma \in [2/3, 4/3], \\ s(\gamma/2 + 1/2) & \text{if } \gamma \in [-4/3, -2/3], \\ c(\gamma + 1/2) & \text{if } \gamma \in [-2/3, -1/3]. \end{cases}$$

3. Then verify that

   1. $\sum_{j \in \mathbf{Z}} |\widehat{\psi}(2^j \gamma)|^2 \equiv 1$, and

   2. $\sum_{j=0}^{\infty} \widehat{\psi}(2^j \gamma)\, \overline{\widehat{\psi}(2^j(\gamma + k))} = 0$ for all odd integers k.

Remarks. (a) The functions $s(x)$ and $c(x)$ can be made as smooth as desired (even up to C^∞). The smoother these functions, the more rapidly decaying will be the wavelet $\psi(x)$. Since $\widehat{\psi}$ is compactly supported, its frequency localization is perfect, and a rapidly decaying $\psi(x)$ gives good time localization.

(b) The seemingly miraculous cancellations that verify condition 2. above can be understood much more easily using the notion of a Multiresolution Analysis (MRA). In fact, the Haar basis and the bandlimited wavelet basis can all be understood in this way.