► **EXERCISE 1.46:** Let the function y have the representation

$$y(t) := \sum_{k=1}^{N} c_k e^{2\pi i s_k t}, \qquad -\infty < t < \infty$$

where $N = 1, 2, \ldots$ and the (not necessarily uniformly spaced) frequencies $s_1 < s_2 < \cdots < s_N$ are known but where $c_1, c_2, \ldots, c_N$ are unknown complex parameters. In this exercise you will develop an analysis equation that can be used to find these coefficients.

Such problems occasionally arise in the natural sciences. For example, the height of sea water in a given harbor is well modeled by a sum of sinusoids that correspond to the earth's rotation (with the sun, moon giving rise to terms with frequencies 2/day, 1.9323/day), to the moon's revolution about the earth, to the earth's revolution about the sun, to the moon's motion out of the plane of the earth's equator, etc., *cf.* R.A.R. Tricker, *Bores, Breakers, Waves, and Wakes*, American Elsevier, New York, 1965, pp. 1–22.

(a) Show that for each $k = 1, 2, \ldots, N$

$$c_k = \lim_{t_1 \to -\infty} \frac{1}{t_2 - t_1} \int_{t=t_1}^{t_2} e^{-2\pi i s_k t} y(t)\,dt = \lim_{t_2 \to +\infty} \frac{1}{t_2 - t_1} \int_{t=t_1}^{t_2} e^{-2\pi i s_k t} y(t)\,dt.$$

In particular, if we have observed $y(t)$ at all times $t \le t_0$, we can use the above analysis equation to find $c_1, c_2, \ldots, c_N$ and then predict $y(t)$ at all times $t > t_0$.

(b) Show that the above trigonometric sum vanishes for $-\infty < x < \infty$ if and only if $c_1 = c_2 = \cdots = c_N = 0$.

(c) Assume that $c_1, c_2, \ldots, c_N$ are all nonzero. Show that y is p-periodic for some $p > 0$, if and only if the products $ps_1, ps_2, \ldots, ps_N$ are all integers. (This will be the case when $s_1, s_2, \ldots, s_N$ are *commensurate*, *i.e.*, s_k/s_ℓ is a rational number for each choice of $k, \ell = 1, 2, \ldots, N$ with $s_k \neq 0$.)

Note. In cases where the frequencies $s_1 < s_2 < \cdots < s_N$ are not commensurate, the trigonometric sum y is not periodic. Such a function is *almost periodic*, however, in the sense that for every choice of $\epsilon > 0$ there are infinitely many ϵ-approximate periods p_n, with

$$|y(x + p_n) - y(x)| < \epsilon, \qquad -\infty < x < \infty, \qquad n = 0, \pm 1, \pm 2, \ldots.$$

These p_n's are more or less uniformly distributed on the real line in the sense that every interval of length B contains at least one of them when $B > 0$ is sufficiently large, *cf.* H. Bohr, *Almost Periodic Functions*, Julius Springer, Berlin, 1933; English translation by H. Cohn, Chelsea, New York, 1947, pp. 32, 80.

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