

MATH 675 – HOMEWORK #5  
DUE 21 MARCH 2007

**Exercise 1.** Exercise 8, page 89 in Stein and Shakarchi.

**Exercise 2.** For each pair of integers  $(j, k)$  define the *dyadic interval*  $I_{j,k}$  by

$$I_{j,k} = [2^{-j}k, 2^{-j}(k+1)],$$

and the *Haar function*  $h_{j,k}(x)$  by

$$h_{j,k} = 2^{j/2}(\chi_{I_{j+1,2k}} - \chi_{I_{j+1,2k+1}})$$

where for any interval  $I$ ,  $\chi_I(x) = 1$  if  $x \in I$  and 0 if  $x \notin I$ . Prove the following.

- (a) For all  $(j, k)$ ,  $I_{j,k} = I_{j+1,2k} \cup I_{j+1,2k+1}$ .
- (b) The system  $\{h_{j,k}\}_{j,k \in \mathbf{Z}}$  is an orthonormal system with respect to the inner product  $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)} dx$ . (Hint: You may use the result of part (a) to show this.)

**Exercise 3.** Consider the *Legendre polynomials*  $\{L_n\}_{n=0}^{\infty}$  defined by the formula at the beginning of Problem 2, p. 95 of Stein and Shakarchi. Prove the following.

- (a) Each  $L_n(x)$  is a polynomial of degree  $n$ . (Hint: Use induction.)
- (b) The system  $\{L_n\}_{n=0}^{\infty}$  is an orthogonal system in the space of Riemann integrable functions on  $[-1, 1]$  with the inner product  $\langle f, g \rangle = \int_{-1}^1 f(x)\overline{g(x)} dx$ . (Hint: Use integration by parts in the strategy of Problem 2(a) of Stein and Shakarchi.)
- (c) For each  $n$ , the monomial  $x^n$  is in the linear span of  $\{L_m\}_{m=0}^n$ . (Hint: Use induction once again.)