8.1. Euclidean Space.

<u>Definition.</u> Let n be a natural number. The set $\mathbb{R}^n$, defined by

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1, x_2, \dots, x_n): x_j \in \mathbb{R}\}$$

Is the Cartesian product of n copies of $\mathbb{R}$. We usually write $\mathbb{x} = (x_1, x_2, \dots, x_n)$.

<u>Remark</u>. (a) $\mathbb{R}^2$ is the Cartesian plane and $\mathbb{R}^3$ is Cartesian 3-space. We say $\mathbb{R}^n$ is Euclidean n-space.

(b) $\mathbb{x} = (x_1, x_2, \dots, x_n) = \mathbb{y} = (y_1, y_2, \dots, y_n)$ if and only if $x_j = y_j$ for all j. The vector $\mathbb{0} = (0,0, \dots, 0)$ is the zero vector or the origin.

(c) So far, $\mathbb{R}^n$ has been defined only as a set, but other structure can be imposed on it.

A. Algebraic structure

$\mathbb{R}^n$ is a *vector space* (see the definition and axioms on p. 59).

B. Geometric Structure

1. Definition. The *dot product* (or *scalar product,* or *inner product*) of $\mathbb{x}, \mathbb{y} \in \mathbb{R}^n$, denoted $\mathbb{x} \cdot \mathbb{y}$ or $\langle \mathbb{x}, \mathbb{y} \rangle$ is given by

$$\mathbb{x} \cdot \mathbb{y} = \langle \mathbb{x}, \mathbb{y} \rangle = \sum_{j=1}^{n} x_j y_j$$

2. The interaction of the algebraic and geometric structure of $\mathbb{R}^n$ is given in Definition 8.1.1 in the book. This definition also gives the defining characteristics of a scalar product.

3. The inner product defines a geometric structure on $\mathbb{R}^n$ because it allows us to define a notion of the *angle between* $\mathbb{x}$ *and* $\mathbb{y}$. More on this later.

C. Topological Structure

1. Definition. The *(Euclidean) norm* of $\mathbb{x} \in \mathbb{R}^n$, denoted $\|\mathbb{x}\|$ or sometimes $\|\mathbb{x}\|_2$ is

$$\|\mathbb{x}\| = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2} = (\langle \mathbb{x}, \mathbb{x} \rangle)^{1/2}$$

2. Remark. (a) $\|\mathbb{x}\|$ is the usual notion of the length of the arrow representing the vector $\mathbb{x} \in \mathbb{R}^2$ or $\mathbb{R}^3$ and generalizes the notion of absolute value on $\mathbb{R}$.

   (b) The norm defines a notion of *distance* by denoting the distance between $\mathbb{x}$ and $\mathbb{y}$ as $\|\mathbb{x} - \mathbb{y}\| = \|\mathbb{y} - \mathbb{x}\|$.

3. Now that we have a notion of distance in $\mathbb{R}^n$, we can talk about convergence of sequences, viz.

   Definition. Let $\mathbb{x}^{(k)}$ be a sequence in $\mathbb{R}^n$. We say that $\mathbb{x}^{(k)} \to \mathbb{x}$ if $\left\|\mathbb{x}^{(k)} - \mathbb{x}\right\| \to 0$ as $k \to \infty$.

4. <u>Theorem</u>.

A sequence $\mathbb{x}^{(k)} = (x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)})$ in $\mathbb{R}^n$ converges to $\mathbb{x} = (x_1, x_2, \dots, x_n)$ in $\mathbb{R}^n$ if and only if for each j, $\lim_{k\to\infty} x_j^{(k)} = x_j$.

Proof:

D. Interaction of topological and geometric structure.

1. <u>Claim</u>. $\|\mathbb{x} - \mathbb{y}\|^2 = \|\mathbb{x}\|^2 + \|\mathbb{y}\|^2 - 2\langle \mathbb{x}, \mathbb{y} \rangle$

2. If $\mathbb{x}, \mathbb{y} \in \mathbb{R}^2$ then the Law of Cosines says that
$$\|\mathbb{x} - \mathbb{y}\|^2 = \|\mathbb{x}\|^2 + \|\mathbb{y}\|^2 - 2\|\mathbb{x}\|\|\mathbb{y}\| \cos\theta$$
where θ is the angle between $\mathbb{x}$ and $\mathbb{y}$.

3. This implies that $\langle \mathbb{x}, \mathbb{y} \rangle = \|\mathbb{x}\|\|\mathbb{y}\| \cos\theta$ and therefore we can *define* the angle between any two vectors in $\mathbb{R}^n$ in this way.

4. Theorem. (Cauchy-Schwarz inequality)
Given $\mathbb{x}, \mathbb{y} \in \mathbb{R}^n$, $|\langle \mathbb{x}, \mathbb{y} \rangle| \leq \|\mathbb{x}\| \|\mathbb{y}\|$ with equality holding if and only if $\mathbb{x}$ and $\mathbb{y}$ are parallel, that is, one is a scalar multiple of the other.

Proof:

5. <u>Theorem</u>. Let $\mathbb{x}, \mathbb{y} \in \mathbb{R}^n$. Then

a. $\|\mathbb{x}\| \geq 0$ with equality holding if and only if $\mathbb{x} = \mathbb{0}$.

b. $\|\alpha\mathbb{x}\| = |\alpha|\|\mathbb{x}\|$ for all $\alpha \in \mathbb{R}$.

c. $\|\mathbb{x} + \mathbb{y}\| \leq \|\mathbb{x}\| + \|\mathbb{y}\|$. (Triangle inequality)

Proof: