MATH 316 – HOMEWORK 9
SOLUTIONS TO SELECTED EXERCISES

Section 12.1, Exercise 4.

(b). By a result in the book we already know that $Vol(E_1 \cup E_2) \leq Vol(E_1) + Vol(E_2)$ always. So we must show that when $Vol(E_1 \cap E_2) = 0$ then $Vol(E_1 \cup E_2) \geq Vol(E_1) + Vol(E_2)$. We will also use the topological facts that $\overline{E_1 \cup E_2} = \overline{E_1} \cup \overline{E_2}$ and that $\overline{E_1 \cap E_2} = \overline{E_1} \cap \overline{E_2}$.

So let $\epsilon > 0$ and choose a grid $\mathcal{G}$ such that $V(E_1 \cap E_2; \mathcal{G}) < \epsilon$. Also note that a rectangle $R \in \mathcal{G}$ satisfies $R \cap \overline{E_1} \neq \emptyset$ and $R \cap \overline{E_2} \neq \emptyset$ if and only if $R \cap (\overline{E_1} \cap \overline{E_1}) \neq \emptyset$. Therefore

$$\begin{aligned} V(E_1 \cup E_2; \mathcal{G}) &= \sum_{\substack{R \in \mathcal{G} \\ R \cap \overline{E_1} \neq \emptyset}} |R| + \sum_{\substack{R \in \mathcal{G} \\ R \cap \overline{E_2} \neq \emptyset}} |R| - \sum_{\substack{R \in \mathcal{G} \\ R \cap (\overline{E_1} \cap \overline{E_2} \neq \emptyset}} |R| \\ &= V(E_1; \mathcal{G}) + V(E_2; \mathcal{G}) - V(E_1 \cap E_2; \mathcal{G}) \\ &\geq V(E_1; \mathcal{G}) + V(E_2; \mathcal{G}) - \epsilon \\ &\geq Vol(E_1) + Vol(E_2) - \epsilon \end{aligned}$$

Taking the infimum of the left side over all such grids $\mathcal{G}$ gives

$$Vol(E_1 \cup E_2) \geq Vol(E_1) + Vol(E_2) - \epsilon$$

Since $\epsilon > 0$ was arbitrary, we have

$$Vol(E_1 \cup E_2) \geq Vol(E_1) + Vol(E_2)$$

as required.

Exercise 5.

(a). Since $E^\circ \subseteq E \subseteq \overline{E}$ it follows that $\overline{E} \setminus E \subseteq \overline{E} \setminus E^\circ = \partial E$ and that $E \setminus E^\circ \subseteq \overline{E} \setminus E^\circ = \partial E$. Since E is a Jordan region, $Vol(\partial E) = 0$. Hence both sets $\overline{E} \setminus E$ and $E \setminus E^\circ$ are subsets of sets with volume zero, so both are Jordan regions with volume zero. Therefore E° and $\overline{E}$ differ from E by sets of volume zero, hence each is a Jordan region (there is a remark in the section that asserts both of the above named facts).

(b). Since $E^\circ \subseteq E \subseteq \overline{E}$ it follows from one of the results in the book that $Vol(E^\circ) \leq Vol(E) \leq Vol(\overline{E})$. But $\overline{E} = E^\circ \cup (\overline{E} \setminus E^\circ)$ Hence again citing a result from the book,

$$Vol(\overline{E}) \leq Vol(E^\circ) + Vol(\overline{E} \setminus E^\circ) = Vol(E^\circ).$$

Therefore

$$Vol(E^\circ) = Vol(E) = Vol(\overline{E})$$

as required.

Section 12.4, Exercise 4.

(a). Let E be the unit ball in $\mathbf{R}^3$ given by $\{(x,y,z)\colon x^2+y^2+z^2 \le 1\}$, and consider the transformation $\varphi(x,y,z) = (ax, by, cz)$. Then $\varphi(E)$ is the ellipsoid given in the problem because if $(u,v,w) \in \varphi(E)$ then $u = ax$, $v = by$, and $w = cz$ for some $(x,y,z) \in E$. Hence

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} + \frac{w^2}{c^2} = x^2 + y^2 + z^2 = 1.$$

Also it is easy to see that $\Delta_\varphi(x,y,z) = abc$. Therefore by the change of variables formula

$$Vol(\varphi(E)) = \int_{\varphi(E)} 1\,dV = \int_E 1\,|\Delta_\varphi(x,y,z)|\,dV = abc\int_E 1\,dV = \frac{4}{3}\pi\,abc$$

as required.

Exercise 6.

By the change of variables formula

$$Vol(f(R_r(x_0))) = \int_{f(B_r(x_0))} 1\,dV = \int_{B_r(x_0)} |\Delta_f(x)|\,dx.$$

Therefore,

$$\frac{Vol(f(R_r(x_0)))}{Vol(B_r(x_0))} = \frac{1}{Vol(B_r(x_0))}\int_{B_r(x_0)} |\Delta_f(x)|\,dx.$$

But by assumption, $\Delta_f(x)$ is continuous on $B_r(x_0)$ for all $r > 0$ sufficiently small. Hence the hypotheses in the result referred to in the hint to this problem (Problem 5, p. 406 I believe) are satisfied and we can conclude that

$$\lim_{r\to 0}\frac{1}{Vol(B_r(x_0))}\int_{B_r(x_0)} |\Delta_f(x)|\,dx = |\Delta_f(x_0)|$$