MATH 316 – HOMEWORK 8
SOLUTIONS TO SELECTED EXERCISES

Section 11.5, Exercise 1.

Define the function $g(s) = f(\mathbf{a} + s\mathbf{u})$ for $s \in \mathbf{R}$. Then g is an ordinary function of a single real variable. The first thing to show is that $g'(s) = D_{\mathbf{u}}(\mathbf{a} + s\mathbf{u})$. To see this we compute

$$\begin{aligned} g'(s) &= \lim_{h \to \mathbf{0}} \frac{g(s+h) - g(s)}{h} \\ &= \lim_{h \to \mathbf{0}} \frac{f(\mathbf{a} + (s+h)\mathbf{u}) - f(\mathbf{a} + s\mathbf{u})}{h} \\ &= \lim_{h \to \mathbf{0}} \frac{f((\mathbf{a} + s\mathbf{u}) + h\mathbf{u}) - f(\mathbf{a} + s\mathbf{u})}{h} \\ &= D_{\mathbf{u}}(\mathbf{a} + s\mathbf{u}) \end{aligned}$$

by the definition of the directional derivative (Def. 11.19) where $\mathbf{a}$ in the definition has been replaced by $\mathbf{a} + s\mathbf{u}$.

Since $g(s)$ is differentiable on $[0, 1]$ (since by the above $g'(s)$ exists for $s \in (0, 1)$, and since the one–sided derivatives exist at the endpoints), it is continuous on $[0, 1]$. Therefore by the Mean Value Theorem there is a $t \in (0, 1)$ such that $g(1) - g(0) = g'(t)(1 - 0) = g'(t)$. Since $g(1) = f(\mathbf{a} + \mathbf{u})$ and $g(0) = f(\mathbf{a})$ and in light of the previous paragraph, this becomes

$$f(\mathbf{a} + \mathbf{u}) - f(\mathbf{a}) = D_{\mathbf{u}}(\mathbf{a} + t\mathbf{u})$$

as required.

Exercise 5.

(a). We can consider $g_1(t) = f(tx + (1 - t)a, y)$ as a composite $f \circ h_1(t)$ where $h_1 : \mathbf{R} \to \mathbf{R}^2$ is given by $h_1(t) = (tx + (1 - t)a, y)$. Therefore by the Chain Rule,

$$\begin{aligned} g_1'(t) &= D(f)(h_1(t))Dh_1(t) \\ &= \begin{bmatrix} f_{x_1}(tx + (1-t)a, y) & f_{x_2}(tx + (1-t)a, y) \end{bmatrix} \begin{bmatrix} d(tx + (1-t)a)/dt \\ d(y)/dt \end{bmatrix} \\ &= \begin{bmatrix} f_{x_1}(tx + (1-t)a, y) & f_{x_2}(tx + (1-t)a, y) \end{bmatrix} \begin{bmatrix} x - a \\ 0 \end{bmatrix} \\ &= f_{x_1}(tx + (1-t)a, y)(x - a) \end{aligned}$$

Similarly if $g_2(t) = f(a, ty + (1 - t)b)$ then

$$g_2'(t) = f_{x_2}(a, ty + (1 - t)b)(y - b).$$

Note that $g(t) = g_1(t)+g_2(t)$, that $g(1) = f(x, y)+f(a, y)$, $g(0) = f(a, y)+f(a, b)$, and finally that

$$g'(t) = f_{x_1}(tx + (1 - t)a, y)(x - a) + f_{x_2}(a, ty + (1 - t)b)(y - b)$$

as required.

(b). Since f is differentiable in the ball $B_r(a, b)$, g is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Therefore by the Mean Value Theorem, there is an $s \in (0, 1)$ such that

$$g(1) - g(0) = g'(s)(1 - 0) = g'(s).$$

But this is the same as

$$f(x, y)+f(a, y)-(f(a, y)+f(a, b)) = f_{x_1}(sx+(1-s)a, y)(x-a)+f_{x_2}(a, sy+(1-s)b)(y-b).$$

Letting $c = sx + (1 - s)a$ and $d = sy + (1 - s)b$ we have that

$$f(x, y) - f(a, b) = f_{x_1}(c, y)(x - a) + f_{x_2}(a, d)(y - b)$$

as required.

Section 11.6, Exercise 1.

(a). In this case, f is a linear transformation with matrix $B = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$. Then clearly f is invertible on all of $\mathbf{R}^2$ by the linear transformation with matrix $B^{-1} = \dfrac{1}{17}\begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix}$. In this case $D(f^{-1})(a, b) = B^{-1}$, that is, it is a constant matrix.

(b). In this case note that $f(0, 0) = (0, 1)$ (in fact, for any integer n, $f(2\pi n, -2\pi n) = (0, 1)$, but we will choose $(0, 0)$.) By the Inverse Function Theorem it is enough to check that $\Delta_f(0, 0) \neq 0$. But

$$\Delta_f(u, v) = \begin{vmatrix} 1 & 1 \\ \cos(u) & -\sin(v) \end{vmatrix} = -(\sin(v) + \cos(u))$$

so that $\Delta_f(0, 0) = -1 \neq 0$.

Exercise 2.

(a). Let $F(x, y, z) = xyz + \sin(x + y + z)$. Since $F(0, 0, 0) = 0$ the Implicit Function Theorem says that we can solve for z in terms of x and y as long as $F_z(0, 0, 0) \neq 0$. But $F_z(x, y, z) = xy + \cos(x + y + z)$ so that $F_z(0, 0, 0) = 1 \neq 0$.

(b). In this case, $F(x, y, z) = x^2 + y^2 + z^2 + (2xy + 3z + 8)^{1/3} - 2$ and as before we can solve for z in terms of x and y as long as $F_z(0, 0, 0) \neq 0$. But $F_z(x, y, z) = 2z + (2xy + 3z + 8)^{-2/3}$ so that $F_z(0, 0, 0) = 1/4 \neq 0$.

Exercise 3.

In this case we define the function $F\colon \mathbf{R}^5 \to \mathbf{R}^3$ by

$$F(x,y,u,v,w) = (u^5 + xv^2 - y + w, u^5 + yu^2 - x + w, w^4 + y^5 - x^4 - 1).$$

Since $F(1,1,1,1,-1) = 0$, the Implicit Function Theorem says that we can solve for u, v, and w in terms of x and y as long as the determinant

$$\frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)} = \begin{vmatrix} \partial F_1/\partial u & \partial F_1/\partial v & \partial F_1/\partial w \\ \partial F_2/\partial u & \partial F_2/\partial v & \partial F_2/\partial w \\ \partial F_3/\partial u & \partial F_3/\partial v & \partial F_3/\partial w \end{vmatrix}$$

does not vanish at the point $(1,1,1,1,-1)$. But

$$\frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)} = \begin{vmatrix} 5u^4 & 2xv & 1 \\ 2yu & 5v^4 & 1 \\ 0 & 0 & 4w^3 \end{vmatrix} = 4w^3(25u^4v^4 - 4xyuv) = -84 \neq 0$$

when evaluated at $(1,1,1,1,-1)$.