MATH 316 – HOMEWORK 7
SOLUTIONS TO SELECTED EXERCISES

Section 11.2, Exercise 8.

Let $\mathbf{a} \in \mathbf{R}^n$. By definition, $DT(\mathbf{a})$ is the unique linear transformation satisfying

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|T(\mathbf{a}+\mathbf{h}) - T(\mathbf{a}) - DT(\mathbf{a})(\mathbf{h})\|}{\|\mathbf{h}\|} = 0.$$

Hence it suffices to show that $DT(\mathbf{a}) = T$ satisfies the above equation. To that end, note that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|T(\mathbf{a}+\mathbf{h}) - T(\mathbf{a}) - T(\mathbf{h})\|}{\|\mathbf{h}\|} = \lim_{\mathbf{h}\to\mathbf{0}} \frac{\|T(\mathbf{a}) + T(\mathbf{h}) - T(\mathbf{a}) - T(\mathbf{h})\|}{\|\mathbf{h}\|} = \lim_{\mathbf{h}\to\mathbf{0}} \frac{0}{\|\mathbf{h}\|} = 0.$$

Exercise 9.

(a). If $\mathbf{u} = \mathbf{e}_k$ then

$$D_{\mathbf{u}}f(\mathbf{a}) = D_{\mathbf{e}_k}f(\mathbf{a}) = \lim_{t\to 0} \frac{f(\mathbf{a}+t\mathbf{e}_k) - f(\mathbf{a})}{t} = \frac{\partial f}{\partial x_k}(\mathbf{a})$$

by the definition of the partial derivative on p. 322.

(b). If $D_{\mathbf{u}}f(\mathbf{a})$ exists for all $\mathbf{u}$ with $\|\mathbf{u}\| = 1$ then in particular it exists for $\mathbf{u} = \mathbf{e}_k$. By part (a), $D_{\mathbf{e}_k}f(\mathbf{a}) = \frac{\partial f}{\partial x_k}(\mathbf{a})$ so the partials exist. To see that the converse does not hold let $f(x, y)$ be as in Example 11.11 and take $\mathbf{u} = (1/\sqrt{2}, 1/\sqrt{2})$. Then

$$D_{\mathbf{u}}f(0,0) = \lim_{t\to 0} \frac{f(1/\sqrt{2}, 1/\sqrt{2}) - f(0,0)}{t} = \lim_{t\to 0} \frac{1}{t}$$

which does not exist.

(c). Let $\mathbf{u} = (u_1, u_2)$. Then

$$D_{\mathbf{u}}f(0,0) = \lim_{t\to 0} \frac{f(tu_1, tu_2) - f(0,0)}{t} = \lim_{t\to 0} \frac{u_1^2 u_2 t^3}{u_1^4 t^4 + u_2^2 t^2} = \lim_{t\to 0} t\, \frac{u_1^2 u_2}{u_1^4 t^2 + u_2^2} = 0.$$

However, note that

$$\lim_{x\to 0} \lim_{y\to 0} \frac{x^2 y}{x^4 + y^2} = \lim_{x\to 0} \frac{0}{x^4} = 0$$

but

$$\lim_{\substack{(x,y)\to(0,0) \\ y=x^2}} \frac{x^2 y}{x^4 + y^2} = \lim_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2}.$$

Therefore the limit does not exist, f is not continuous at $(0, 0)$, and all the more so it is not differentiable there.

Section 11.4, Exercise 3.

Defining $h\colon \mathbf{R}^2 \to \mathbf{R}$ with $(x,y) \mapsto xy$, we can write $u = f \circ h$ and by the Chain Rule,

$$Du(x,y) = Df(h(x,y))Dh(x,y).$$

Expanding, this becomes

$$\begin{bmatrix} \partial u/\partial x & \partial u/\partial y \end{bmatrix} = f'(xy) \begin{bmatrix} \partial h/\partial x & \partial h/\partial y \end{bmatrix} = f'(xy) \begin{bmatrix} y & x \end{bmatrix}.$$

Therefore

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = \begin{bmatrix} \partial u/\partial x & \partial u/\partial y \end{bmatrix} \begin{bmatrix} x \\ -y \end{bmatrix} = f'(xy) \begin{bmatrix} y & x \end{bmatrix} \begin{bmatrix} x \\ -y \end{bmatrix} = 0.$$

If $v(x,y) = f(x-y) + g(x+y)$ then

$$\frac{\partial v}{\partial x} = f'(x-y) + g'(x+y) \quad \text{and} \quad \frac{\partial v}{\partial y} = -f'(x-y) + g'(x+y).$$

Taking a second derivative we get

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = f''(x-y) + g''(x+y)$$

and

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = -(-f''(x-y)) + g''(x+y) = f''(x-y) + g''(x+y) = \frac{\partial^2 v}{\partial x^2}.$$

Exercise 9.

Defining $h\colon \mathbf{R}^2 \to \mathbf{R}^2$ with $(r,\theta) \mapsto (r\cos\theta, r\sin\theta)$, we can write $u = f \circ h$ and $u = g \circ h$. By the Chain Rule,

$$Du(r,\theta) = DF(h(r,\theta))Dh(r,\theta)$$

and expanding we get

$$\begin{bmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial(r\cos\theta)}{\partial r} & \frac{\partial(r\cos\theta)}{\partial \theta} \\ \frac{\partial(r\sin\theta)}{\partial r} & \frac{\partial(r\sin\theta)}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

and similarly

$$\begin{bmatrix} \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}.$$

By the Cauchy–Riemann equations we can write

$$\frac{\partial u}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta = \frac{\partial g}{\partial y}\cos\theta - \frac{\partial g}{\partial x}\sin\theta = \frac{1}{r}\left(-r\frac{\partial g}{\partial x}\sin\theta + r\frac{\partial g}{\partial y}\cos\theta\right) = \frac{1}{r}\frac{\partial v}{\partial \theta}.$$

The other identity follows similarly.