MATH 316 – HOMEWORK 6
SOLUTIONS TO SELECTED EXERCISES

Section 9.1, Exercise 3.

$\mathbf{x}_k \to \mathbf{a}$ if and only if $\|\mathbf{x}_k\| \to 0$ as $k \to \infty$ and $\{\mathbf{y}_k\}$ bounded means that there is an $M \in \mathbf{R}$ such that $\|\mathbf{y}_k\| \leq M$ for all $k \in \mathbf{N}$. By the Cauchy–Schwarz inequality,

$$0 \leq |\mathbf{x}_k \cdot \mathbf{y}_k| \leq \|\mathbf{x}_k\|\|\mathbf{y}_k\| \leq M\,\|\mathbf{x}_k\|.$$

Since $M\,\|\mathbf{x}_k\| \to 0$ as $k \to \infty$, the Squeeze Theorem says that $\mathbf{x}_k \cdot \mathbf{y}_k \to 0$ as $k \to \infty$.

Exercise 5.

(i). Suppose that $\mathbf{x}_k \to \mathbf{a}$ and $\mathbf{x}_k \to \mathbf{b}$ as $k \to \infty$. Given $\epsilon > 0$ there is a $k_0 \in \mathbf{N}$ such that $\|\mathbf{x}_{k_0} - \mathbf{a}\| < \epsilon/2$ and $\|\mathbf{x}_{k_0} - \mathbf{b}\| < \epsilon/2$. By the triangle inequality

$$\|\mathbf{a} - \mathbf{b}\| \leq \|\mathbf{a} - \mathbf{x}_k\| + \|\mathbf{x}_k - \mathbf{b}\| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, $\|\mathbf{a} - \mathbf{b}\| = 0$ or $\mathbf{a} = \mathbf{b}$.

(ii). Suppose that $\mathbf{x}_k \to \mathbf{a}$ as $k \to \infty$, let $\{\mathbf{x}_{k_j}\}$ be given, and let $\epsilon > 0$. Since $\mathbf{x}_k \to \mathbf{a}$ there is an N such that if $k \leq N$ then $\|\mathbf{x}_k - \mathbf{a}\| < \epsilon$. Since $\{\mathbf{x}_{k_j}\}$ is a subsequence, $k_j \geq j$ for all j and hence if $j \geq N$, $k_j \geq j \geq N$ so that $\|\mathbf{x}_{k_j} - \mathbf{a}\| < \epsilon$ as well. Therefore, $\mathbf{x}_{k_j} \to \mathbf{a}$ as $j \to \infty$.

Exercise 6.

(i). Suppose that $\mathbf{x}_k \to \mathbf{a}$ as $k \to \infty$. By the triangle inequality,

$$0 \leq |\|\mathbf{x}_k\| - \|\mathbf{a}\|| \leq \|\mathbf{x}_k - \mathbf{a}\|$$

so that by the Squeeze Theorem, $\|\mathbf{x}_k\| \to \|\mathbf{a}\|$. Since $\{\|\mathbf{x}_k\|\}_{k \in \mathbf{N}}$ is a convergent sequence of real numbers, it is bounded, hence $\{\mathbf{x}_k\}$ is bounded.

(ii). Suppose that $\mathbf{x}_k \to \mathbf{a}$ as $k \to \infty$ and let $\epsilon > 0$. There is an $N \in \mathbf{N}$ such that $k \geq N$ implies that $\|\mathbf{x}_k - \mathbf{a}\| < \epsilon/2$. If n, $m \geq N$ then by the triangle inequality,

$$\|\mathbf{x}_n - \mathbf{x}_m\| \leq \|\mathbf{x}_k - \mathbf{a}\| + \|\mathbf{a} - \mathbf{x}_m\| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence $\{\mathbf{x}_k\}$ is Cauchy.

Exercise 10.

(a). ($\Longrightarrow$) Suppose that $\mathbf{a}$ is a cluster point of E and let $r > 0$. Since $E \cap B_r(\mathbf{a})$ contains infinitely many points it contains at least two points, at least one of which is not $\mathbf{a}$. Hence $E \cap B_r(\mathbf{a}) \setminus \{\mathbf{a}\}$ is not empty.

($\Longleftarrow$) Let $r > 0$. Since $E \cap B_r(\mathbf{a}) \setminus \{\mathbf{a}\}$ is not empty there is a point $\mathbf{x}_1 \in E \cap B_r(\mathbf{a}) \setminus \{\mathbf{a}\}$. Let $r_1 = \|\mathbf{x}_1 - \mathbf{a}\|$. Since $\mathbf{x}_1 \neq \mathbf{a}$, $r_1 > 0$ and by hypothesis there is a point $\mathbf{x}_2 \in E \cap B_{r_1}(\mathbf{a}) \setminus \{\mathbf{a}\}$. Clearly $\mathbf{x}_2 \neq \mathbf{x}_1$ since $\|\mathbf{x}_2 - \mathbf{a}\| < r_1 = \|\mathbf{x}_1 - \mathbf{a}\|$ and also $\mathbf{x}_2 \neq \mathbf{a}$. Letting $r_2 = \|\mathbf{x}_2 - \mathbf{a}\|$, $r_2 > 0$ and by hypothesis we can choose x_3 distinct from $\mathbf{x}_2$ and $\mathbf{x}_1$ in $E \cap B_{r_2}(\mathbf{a}) \setminus \{\mathbf{a}\}$. Continuing in this fashion we can define an infinite sequence of *distinct* points $\{\mathbf{x}_k\} \subseteq E \cap B_r(\mathbf{a})$ as required.

An alternate proof for this direction is the following. Suppose that for some $r > 0$, $E \cap B_r(\mathbf{a})$ and hence also $E \cap B_r(\mathbf{a}) \setminus \{\mathbf{a}\}$ is finite. If we enumerate the set as $\{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ then for each $1 \leq j \leq N$, $\|\mathbf{x}_j - \mathbf{a}\| > 0$. Let $r_0 = \min\{\|\mathbf{x}_j - \mathbf{a}\| \colon 1 \leq j \leq N\}$. Then $r_0 > 0$ and $E \cap B_{r_0}(\mathbf{a}) \setminus \{\mathbf{a}\}$ is empty. (Note: This is a proof of this implication by contrapositive.)

(b). If E is a bounded infinite subset of $\mathbf{R}$ then there exists an infinite sequence $\{\mathbf{x}_k\}_{k \in \mathbf{N}}$ of distinct points in E. By the Bolzano–Weierstrass Theorem, $\{\mathbf{x}_k\}$ has a convergent subsequence $\{\mathbf{x}_{k_j}\}_{j \in \mathbf{N}}$ converging to some $\mathbf{a} \in \mathbf{R}^n$. Since the $\mathbf{x}_{k_j}$ are distinct points, $\mathbf{a}$ is a cluster point of $\{\mathbf{x}_{k_j}\}_{j \in \mathbf{N}}$ and hence also of E.

Section 9.2, Exercise 2.

(a).

$$\lim_{x \to 0} \lim_{y \to 0} \frac{\sin(x)\sin(y)}{x^2 + y^2} = \lim_{x \to 0} \frac{\sin(x)\sin(0)}{x^2} = 0.$$

$$\lim_{y \to 0} \lim_{x \to 0} \frac{\sin(x)\sin(y)}{x^2 + y^2} = \lim_{y \to 0} \frac{\sin(0)\sin(y)}{y^2} = 0.$$

Letting $y = x$ we have

$$\lim_{\substack{(x,y) \to (0,0) \\ y = x}} \frac{\sin(x)\sin(y)}{x^2 + y^2} = \lim_{x \to 0} \frac{\sin^2(x)}{2x^2} = \frac{1}{2}.$$

Therefore the limit does not exist.

(b).

$$\lim_{x \to 0} \lim_{y \to 0} \frac{x^2 + y^4}{x^2 + 2y^4} = \lim_{x \to 0} \frac{x^2}{x^2} = 1.$$

$$\lim_{y \to 0} \lim_{x \to 0} \frac{x^2 + y^4}{x^2 + 2y^4} = \lim_{y \to 0} \frac{y^4}{2y^4} = \frac{1}{2}.$$

Therefore the limit does not exist.

(c).

$$\lim_{x \to 0} \lim_{y \to 0} \frac{x - y}{(x^2 + y^2)^\alpha} = \lim_{x \to 0} \frac{x}{x^{2\alpha}} = \lim_{x \to 0} x^{1 - 2\alpha} = 0$$

since $1-2\alpha > 0$. Similarly,

$$\lim_{y\to 0}\lim_{x\to 0}\frac{x-y}{(x^2+y^2)^\alpha} = \lim_{y\to 0} -y^{1-2\alpha} = 0.$$

To see that $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x-y}{(x^2+y^2)^\alpha} = 0$, note that we have the estimates

$$\left|\frac{x}{(x^2+y^2)^\alpha}\right| = |x|^{1-2\alpha}\left|\frac{x^{2\alpha}}{(x^2+y^2)^\alpha}\right| = |x|^{1-2\alpha}\left(\frac{x^2}{x^2+y^2}\right)^\alpha \le |x|^{1-2\alpha}$$

since $x^2/(x^2+y^2) \le 1$ for all $(x,y) \ne (0,0)$. Similarly

$$\left|\frac{y}{(x^2+y^2)^\alpha}\right| \le |y|^{1-2\alpha}$$

for all $(x,y) \ne (0,0)$. Now given $\epsilon > 0$ choose $\delta > 0$ so that $\delta < (\epsilon/2)^{1/(1-2\alpha)}$. If $(x^2+y^2)^{1/2} < \delta$ then also $|x|$ and $|y| < \delta$ and $|x|^{1-2\alpha}$ and $|y|^{1-2\alpha} < \epsilon/2$. Therefore

$$\left|\frac{x-y}{(x^2+y^2)^\alpha}\right| \le \left|\frac{x}{(x^2+y^2)^\alpha}\right| + \left|\frac{y}{(x^2+y^2)^\alpha}\right| \le |x|^{1-2\alpha} + |y|^{1-2\alpha} \le \epsilon/2 + \epsilon/2 = \epsilon.$$

Exercise 3.

(a). To see that $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^3-y^3}{x^2+y^2} = 0$, note that we have the estimates

$$\left|\frac{x^3-y^3}{x^2+y^2}\right| \le |x|\left|\frac{x^2}{x^2+y^2}\right| + |y|\left|\frac{y^2}{x^2+y^2}\right| \le |x|+|y| \le \sqrt{2}(x^2+y^2)^{1/2}$$

where the final inequality follows from the Cauchy–Schwarz inequality. Hence given $\epsilon > 0$ choose $\delta < \epsilon/\sqrt{2}$. Then if $(x^2+y^2)^{1/2} < \delta$, $\displaystyle\left|\frac{x^3-y^3}{x^2+y^2}\right| \le \sqrt{2}(x^2+y^2)^{1/2} < \epsilon$.

(b). To see that $\displaystyle\lim_{(x,y)\to(0,0)}\frac{|x|^\alpha y^4}{x^2+y^4} = 0$, note that we have the estimates

$$\left|\frac{|x|^\alpha y^4}{x^2+y^4}\right| \le |x|^\alpha\left|\frac{y^4}{x^2+y^4}\right| \le |x|^\alpha \le (x^2+y^2)^{\alpha/2}$$

since $y^4/(x^2+y^4) \le 1$ for all $(x,y) \ne (0,0)$ and since $|x| \le (x^2+y^2)^{1/2}$. Hence given $\epsilon > 0$ choose $\delta < \epsilon^{1/\alpha}$. Then if $(x^2+y^2)^{1/2} < \delta$, $\displaystyle\left|\frac{|x|^\alpha y^4}{x^2+y^4}\right| \le (x^2+y^2)^{\alpha/2} < \epsilon$.