MATH 316 – HOMEWORK 5
SOLUTIONS TO SELECTED EXERCISES

Section 8.1, Exercise 6.

In order to prove that $\sum_{k=1}^{\infty} a_k b_k$ converges absolutely it is enough to show that the partial sums $\sum_{k=1}^{n} |a_k b_k|$ are bounded independently of n, that is, that there is a number $M \in \mathbf{R}$ such that for all n, $\sum_{k=1}^{n} |a_k b_k| \le M$.

Given n and applying the Cauchy-Schwarz inequality to the sum $\sum_{k=1}^{n} |a_k b_k|$ we have

$$\sum_{k=1}^{n} |a_k b_k| \le \left(\sum_{k=1}^{n} |a_k|^2\right)^{1/2} \left(\sum_{k=1}^{n} |b_k|^2\right)^{1/2} = \left(\sum_{k=1}^{n} a_k^2\right)^{1/2} \left(\sum_{k=1}^{n} b_k^2\right)^{1/2}$$

since the a_k and b_k are real numbers. Since for all k, $a_k^2 \ge 0$ and $b_k^2 \ge 0$,

$$\left(\sum_{k=1}^{n} a_k^2\right)^{1/2} \left(\sum_{k=1}^{n} b_k^2\right)^{1/2} \le \left(\sum_{k=1}^{\infty} a_k^2\right)^{1/2} \left(\sum_{k=1}^{\infty} b_k^2\right)^{1/2} = M < \infty$$

by the hypothesis of the problem. This establishes a bound on the partial sums mentioned above and completes the proof.

Section 8.2, Exercise 1.

(a). The normal vector for such a plane would be a vector perpendicular to both $(0, 1, 3)$ (the vector represented by the arrow from the point $(1, 1, 0)$ to the point $(1, 2, 3)$) and the vector $(-2, 1, -3)$ (the vector represented by the arrow from the point $(1, 1, 0)$ to the point $(-1, 2, -3)$). Such a vector is given by $(3, 3, -1)$ (this can be found by computing a cross-product or by solving a linear system of two equations in 3 unknowns, namely $\begin{bmatrix} -2 & 1 & -3 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$). This leads to the equation $3x + 3y - z = 6$ for the plane.

(b). Solution is arrived at similarly to (a). The plane is given by the equation $-x + y = 3$.

Exercise 3.

As before we require a vector perpendicular to the vectors $(1, 1, 0, 0)$, $(-1, 1, 1, 0)$, and $(-1, -4, 0, 1)$ (these are the vectors represented by the arrows from $(1, 0, 0, 0)$ to $(2, 1, 0, 0)$, $(0, 1, 1, 0)$, and $(0, 4, 0, 1)$ respectively). Setting up the required linear system, namely, $\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, we obtain $(1, -1, 2, 5)$ as a solution. This leads to the equation $x - y + 2z + 5w = 1$ for the hyperplane.

Exercise 5.

(a). By linearity of T,

$$T(0,1) = \frac{1}{2}T(0,2) = (2,0,1/2),$$

$$T(1,0) = T(1,1) - T(1,0) = (3,\pi,0) - (2,0,1/2) = (1,\pi,-1/2).$$

Hence the matrix representation of T is the 3×2 matrix $\begin{bmatrix} 1 & 2 \\ \pi & 0 \\ -1/2 & 1/2 \end{bmatrix}$.

(b). By linearity of T,

$$T(1,0,0) = T(0,-1,1) + T(1,1,-1) = (1,0) + (4,7) = (5,7),$$

$$T(0,1,0) = T(1,1,0) - T(0,-1,1) - T(1,1,-1) = (e,\pi) - (1,0) - (4,7) = (e-5,\pi-7),$$

$$T(0,0,1) = T(1,1,0) - T(1,1,-1) = (e,\pi) - (4,7) = (e-4,\pi-7).$$

Hence the matrix representation of T is the 2×3 matrix $\begin{bmatrix} 5 & e-5 & e-4 \\ 7 & \pi-7 & \pi-7 \end{bmatrix}$.
The particular linear combinations used above were arrived at by observing that

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & -1 & -1 \end{bmatrix}.$$