MATH 316 – HOMEWORK 4
SOLUTIONS TO SELECTED EXERCISES

Section 7.3, Exercise 1.

(b). Let $a_n = [(-1)^n + 3]^n$. Then $|a_n|^{1/n} = (-1)^n + 3 = 2$ if n is odd and 4 if n is even. Therefore $\limsup_{n\to\infty} |a_n|^{1/n} = 4$ and so the radius of convergence $R = 1/4$ and the power series converges absolutely on $(3/4, 5/4)$. Checking the endpoints we have that if $x = 3/4$ then $(x-1)^k = (-1/4)^k$ so that $((-1)^n + 3)^n(-1/4)^n = [((-1)^{n+1} - 3)/4]^n = (-1/2)^n$ if n is odd and $(-1)^n$ if n is even. If $x = 5/4$ then $(x-1)^k = (1/4)^k$ so that $((-1)^n + 3)^n(1/4)^n = [((-1)^n + 3)/4]^n = 1/2^n$ if n is odd and 1 if n is even. In both cases the terms of the series do not converge to zero hence the series diverges. Therefore the interval of convergence is $(3/4, 5/4)$.

(c). Let $a_k = \log[(k+1)/k]$. Then applying L'Hopital's rule,

$$\lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right| = \lim_{k\to\infty}\frac{\log[(k+1)/k]}{\log[(k+2)/(k+1)]} = \lim_{k\to\infty}\frac{\frac{1}{x+1} - \frac{1}{x}}{\frac{1}{x+2} - \frac{1}{x+1}} = \lim_{k\to\infty}\frac{x+2}{x} = 1.$$

Hence the radius of convergence is $R = 1$ and the series converges absolutely on the interval $(-1, 1)$. Checking the series for $x = -1$, we note that $\sum_{k=1}^{\infty}\log\left(\frac{k+1}{k}\right)(-1)^k$ converges by the Alternating Series Test since $\lim_{k\to\infty}\log[(k+1)/k] = \log(1) = 0$ and furthermore that $\log[(k+1)/k]$ is decreasing in k (this follows since $(d/dx)\log[(x+1)/x] = 1/(x+1) - 1/x < 0$ for $x \geq 1$). If $x = 1$ then the series $\sum_{k=1}^{\infty}\log\left(\frac{k+1}{k}\right)$ diverges by the Limit Comparison Test since, using L'Hopital's Rule,

$$\lim_{k\to\infty}\frac{\log[(k+1)/k]}{1/k} = \lim_{k\to\infty}\frac{\frac{1}{k+1} - \frac{1}{k}}{-1/k^2} = \lim_{k\to\infty}\frac{k}{k+1} = 1$$

and since $\sum_k (1/k)$ diverges. Hence the interval of convergence is $[-1, 1)$.

(d). Consider the series $\sum_{k=1}^{\infty}\frac{(1)(3)\cdots(2k-1)}{(k+1)!}t^k$. Applying the Ratio Test we find that

$$\lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right| = \lim_{k\to\infty}\frac{k+2}{2k+1} = 1/2.$$

Hence the above series has radius of convergence $R = 1/2$ and converges absolutely for $|t| < 1/2$. Hence the series given in the problem converge absolutely for $|x^2| < 1/2$ or $|x| < 1/\sqrt{2}$. Inserting the endpoint $x = 1/\sqrt{2}$ gives the series $\sum_{k=1}^{\infty}\frac{(1)(3)\cdots(2k-1)}{2^k(k+1)!}$ and inserting the endpoint $x = -1/\sqrt{2}$ gives the series

$\sum_{k=1}^{\infty}(-1)^k \dfrac{(1)(3)\cdots(2k-1)}{2^k(k+1)!}$. In either case,

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{2k+1}{2k+4} = 1 - \frac{3}{2k+4} = 1 - \frac{(3/2)}{k+2}.$$

Hence in each case the series converges by Raabe's Test and the interval of convergence is $[-1/\sqrt{2}, 1/\sqrt{2}]$.

Exercise 2.

(a). $\sum_{k=1}^{\infty} 3x^{3k-1} = \dfrac{3}{x}\sum_{k=1}^{\infty}(x^3)^k = \dfrac{3}{x}\dfrac{x^3}{1-x^3} = \dfrac{3x^2}{1-x^3}$ valid on $(-1,1)$.

(b).

$$\begin{aligned}
\sum_{k=2}^{\infty} kx^{k-2} &= \frac{1}{x^2}\sum_{k=2}^{\infty} kx^k \\
&= \frac{1}{x^2}\left(\sum_{k=1}^{\infty} kx^k - x\right) \\
&= \frac{1}{x^2}\left(\frac{x}{(1-x)^2} - x\right) \\
&= \frac{1}{x^2}\left(\frac{x - x(1-x)^2}{(1-x)^2}\right) \\
&= \frac{1}{x^2}\frac{2x^2 - x^3}{(1-x)^2} = \frac{2-x}{(1-x)^2}
\end{aligned}$$

where we have used Example 7.36. This identity is valid on $(-1,1)$.

Another approach to the summation is the following.

$$\begin{aligned}
\sum_{k=2}^{\infty} kx^{k-2} &= \sum_{k=0}^{\infty}(k+2)\,x^k \\
&= \sum_{k=0}^{\infty} kx^k + 2\sum_{k=0}^{\infty} x^k \\
&= \sum_{k=1}^{\infty} kx^k + \frac{2}{(1-x)} \\
&= \frac{x}{(1-x)^2} + \frac{2}{(1-x)} = \frac{2-x}{(1-x)^2}
\end{aligned}$$

where we have used Example 7.36 in the last line and in the third line the fact that $\sum_{k=0}^{\infty} kx^k = \sum_{k=1}^{\infty} kx^k$ because the $k=0$ term in the first sum is zero.

(c).

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{2k}{k+1}(1-x)^k &= \sum_{k=1}^{\infty} \left(2 - \frac{2}{k+1}\right)(1-x)^k \\
&= 2\sum_{k=1}^{\infty}(1-x)^k - 2\sum_{k=1}^{\infty}\frac{(1-x)^k}{k+1} \\
&= 2\sum_{k=0}^{\infty}(1-x)^k - 2\sum_{k=0}^{\infty}\frac{(1-x)^k}{k+1} \\
&= \frac{2}{1-(1-x)} + \frac{2\log(1-(1-x))}{1-x} \\
&= \frac{2}{x} + \frac{2\log(x)}{1-x}.
\end{aligned}$$

Here we have used Example 7.37. Also note that in the third line we have used the fact that $\sum_{k=0}^{\infty}(1-x)^k = 1+\sum_{k=1}^{\infty}(1-x)^k$ and that $\sum_{k=0}^{\infty}\frac{(1-x)^k}{k+1} = 1+\sum_{k=1}^{\infty}\frac{(1-x)^k}{k+1}$. Since we are subtracting those sums, 1 can be added to each without changing their difference. This identity is valid on $(0,2)$ if we take $\log(x)/(1-x)$ to have the value 0 at $x=1$.

Exercise 7.

We will prove first that $\displaystyle\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$ for all $x \in \mathbf{R}$. If we can show this then Theorem 7.48 shows that in fact f is analytic on all of $\mathbf{R}$.

The proof of the first statement reduces to showing that the sequence of remainders $R_n(x) = R_n^{0,f}(x)$ converges to zero for all $x \in \mathbf{R}$ as $n \to \infty$. But by Lagrange's Formula,

$$R_{n+1}^{0,f}(x) = \frac{1}{n!}\int_0^x (x-t)^n f^{(n+1)}(t)\,dt = \frac{1}{n!}\int_0^x u^n\, f^{(n+1)}(x-u)\,du$$

where we have made the substitution $u = x - t$ in the integral. By hypothesis, with a replaced by x,

$$\lim_{n\to\infty} R_{n+1}^{0,f}(x) = \lim_{n\to\infty}\frac{1}{n!}\int_0^x u^n\, f^{(n+1)}(x-u)\,du = 0.$$

Exercise 8.

(a). First note that $\displaystyle e^{x^2} = \sum_{k=0}^{\infty}\frac{(x^2)^k}{k!} = \sum_{k=0}^{\infty}\frac{x^{2k}}{k!}$ and hence that

$$e^{x^2} - \sum_{k=0}^{n-1}\frac{x^{2k}}{k!} = R_n(x) = \frac{e^c}{n!}x^{2n}$$

by Taylor's formula for some c between 0 and x. Therefore,

$$\begin{aligned}\int_0^1 \left(e^{x^2} - \sum_{k=0}^{n-1} \frac{x^{2k}}{k!}\right) dx &= \int_0^1 e^{x^2}\,dx - \sum_{k=0}^{n-1} \frac{1}{k!}\int_0^1 x^{2k}\,dx \\ &= \int_0^1 e^{x^2}\,dx - \sum_{k=0}^{n-1} \frac{1}{(2k+1)k!} \\ &= \int_0^1 R_n(x)\,dx.\end{aligned}$$

Now,

$$\left|\int_0^1 R_n(x)\,dx\right| \le \int_0^1 |R_n(x)|\,dx = \int_0^1 \frac{e^c}{n!}x^{2n}\,dx \le \frac{e}{n!}\int_0^1 x^{2n}\,dx = \frac{e}{(2n+1)n!}.$$

Now, if $n \ge 1$ then $1/(2n+1) \le 1/3 < 1$ and certainly $e/(2n+1) < 3$ since $e < 3$. Hence $\left|\int_0^1 R_n(x)\,dx\right| < (3/n!)$.

Another way to do this is the following. Since we can integrate the power series for e^{x^2} term-by-term we have that

$$\int_0^1 e^{x^2}\,dx = \sum_{k=0}^{\infty} \frac{1}{k!}\int_0^1 x^{2k}\,dx = \sum_{k=0}^{\infty} \frac{1}{(2k+1)k!}.$$

Therefore,

$$\begin{aligned}\int_0^1 e^{x^2}\,dx &- \sum_{k=0}^{n-1} \frac{1}{(2k+1)k!} = \sum_{k=n}^{\infty} \frac{1}{(2k+1)k!} \\ &= \frac{1}{2n+1}\frac{1}{n!} + \frac{1}{2n+3}\frac{1}{(n+1)!} + \cdots \\ &= \frac{1}{2n+1}\frac{1}{n!}\left[1 + \frac{2n+1}{2n+3}\frac{1}{n+1} + \frac{2n+1}{2n+5}\frac{1}{(n+1)(n+2)} + \cdots\right] \\ &\le \frac{1}{2n+1}\frac{1}{n!}\left[1 + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \cdots\right] \\ &\le \frac{1}{2n+1}\frac{1}{n!}\left[1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right] \\ &= \frac{1}{2n+1}\frac{1}{n!}\frac{1}{1-\frac{1}{n+1}} \\ &= \frac{1}{2n+1}\frac{1}{n!}\frac{n+1}{n} < (2/3)\frac{1}{n!} < \frac{3}{n!}\end{aligned}$$

since if $n \ge 1$, $1/(2n+1) < 1/3$ and $(n+1)/n < 2$.

A third way to look at it gives the precise answer in the book. Since $|R_n(x)| = (e^c/n!)|x|^{2n} \le e/n! < 3/n!$, for $x \in (0,1)$, we have from before that

$$\left|\int_0^1 e^{x^2}\,dx - \sum_{k=0}^{n-1} \frac{1}{(2k+1)k!}\right| \le \int_0^1 \left|e^{x^2} - \sum_{k=0}^{n-1} \frac{1}{k!}x^{2k}\right| dx = \int_0^1 |R_n(x)|\,dx \le \int_0^1 \frac{3}{n!}\,dx = \frac{3}{n!}.$$