MATH 316 – HOMEWORK 3
SOLUTIONS TO SELECTED EXERCISES

Section 7.1, Exercise 1.

(a). Let $R = \max(|a|, |b|)$. Then $[a, b] \subseteq [-R, R]$ and for all $x \in [a, b]$, $|x/n| \leq R/n$. Given $\epsilon > 0$, choose $N \geq R/\epsilon$. If $n \geq N$ then $|x/n| \leq R/n < \epsilon$ for all $x \in [a, b]$, and $x/n \to 0$ uniformly on $[a, b]$.

(b). Given $x \in (0, 1)$, $1/x \in \mathbf{R}$ and since $1/n \to 0$ as $n \to \infty$, $1/(nx) = (1/x)(1/n) \to 0$ as $n \to \infty$. Hence $1/(nx) \to 0$ pointwise on $(0, 1)$ as $n \to \infty$.

Note that for all $n \in \mathbf{N}$, $\sup_{x\in(0,1)} |1/(nx)| \geq 1$ since if $x = 1/n$ then $1/(nx) = 1/[(1/n)(n)] = 1$. This shows that $1/(nx)$ does not converge to zero uniformly on $(0, 1)$ because if it did then $\sup_{x\in(0,1)} |1/(nx)| \to 0$ as $n \to \infty$.

Exercise 5.

We will first show that under the given hypotheses, $f(x)$ must be bounded. Since each $f_n(x)$ is bounded on E, for each n there is an $M_n \in \mathbf{R}$ such that $|f_n(x)| \leq M_n$ for all $x \in E$. Since $f_n \to f$ uniformly on E, there is an N such that if $n \geq N$ then $|f(x) - f_n(x)| < 1$ for all $x \in E$. Then for all $x \in E$,

$$|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| < 1 + M_N < \infty$$

so that $f(x)$ is bounded.

To show that $\{f_n\}$ is uniformly bounded, note that for all $x \in E$, and $n \geq N$,

$$|f_n(x)| \leq |f_n(x) - f(x)| + |f(x)| < 1 + \sup_{t\in E} |f(t)| < \infty.$$

Therefore for all $n \in \mathbf{N}$, and all $x \in E$,

$$|f_n(x)| \leq \max(M_1,\ M_2,\ \ldots,\ M_{N-1},\ 1 + \sup_{t\in E} |f(t)|).$$

Exercise 6.

Let $\epsilon > 0$. In order to show that f is uniformly continuous on E, we must find a δ such that whenever $x, y \in E$ and $|x-y| < \delta$ then $|f(x)-f(y)| < \epsilon$. Since $f_n \to f$ uniformly we can choose N so large that if $n \geq N$ then $|f(x) - f_n(x)| < \epsilon/3$ for all $x \in E$. Since $f_N(x)$ is uniformly continuous on E, choose $\delta > 0$ so that if $x, y \in E$ and $|x - y| < \delta$ then $|f_N(x) - f_N(y)| < \epsilon/3$. Then whenever $x, y \in E$ and $|x - y| < \delta$,

$$|f(x)-f(y)| \leq |f(x)-f_N(x)|+|f_N(x)-f_N(y)|+|f_N(y)-f(y)| < \epsilon/3+\epsilon/3+\epsilon/3 = \epsilon.$$

Exercise 10.

By way of some perspective on this problem, note that if we replace the number b with $f(x)$ and the numbers b_k with $f_k(x)$ in Exercise 7(b), p. 159, we have that

$$\frac{f_1(x) + f_2(x) + \cdots + f_n(x)}{n} \to f(x)$$

as $n \to \infty$ for each $x \in E$. This says that if $f_k \to f$ *pointwise* on E, then the averages of $\{f_k\}$ also converge *pointwise* to f. The core of the problem is to show that if we replace *pointwise* with *uniformly*, the result still holds.

To this end, note that making the replacements as above, Exercise 7(a), p. 159, says that if there are M, $N \in \mathbf{N}$ such that $|f(x) - f_k(x)| \le M$ for all $k \ge N$ and for all $x \in E$, then for all $x \in E$ and all $n > N$,

$$\left| n\, f(x) - \sum_{k=1}^{n} f_k(x) \right| \le \sum_{k=1}^{N} |f(x) - f_k(x)| + M(n - N).$$

Now we proceed as in the proof of Exercise 7(b) with all necessary changes being made. Let $\epsilon > 0$ and, since $f_k \to f$ uniformly on E, choose N so large that if $k \ge N$ then $|f(x) - f_k(x)| < \epsilon/2$ for all $x \in E$. Then for all $n > N$,

$$\begin{aligned}
\left| f(x) - \frac{f_1(x) + \cdots + f_n(x)}{n} \right| &= \frac{1}{n}\left| n\, f(x) - \sum_{k=1}^{n} f_k(x) \right| \\
&\le \frac{1}{n} \sum_{k=1}^{N} |f(x) - f_k(x)| + \frac{\epsilon}{2} \frac{(n - N)}{n} \\
&\le \frac{1}{n} \sum_{k=1}^{N} |f(x) - f_k(x)| + \frac{\epsilon}{2}
\end{aligned}$$

since $(n - N)/n < 1$ for all $n > N$.

Since each $f_k(x)$ is bounded on E, Problem 5 says that $f(x)$ is also bounded on E. Suppose then that for each $k \in \mathbf{N}$, $|f_k(x)| \le M_k$ and that $|f(x)| \le M$ for all $x \in E$. Then

$$\left| \sum_{k=1}^{N} |f(x) - f_k(x)| \right| \le \sum_{k=1}^{N} (|f(x)| + |f_k(x)|) \le \sum_{k=1}^{N} (M + M_k) = NM + \sum_{k=1}^{N} M_k$$

which number is independent of $x \in E$ and n.

Now choose N_0 so large that if $n \ge N_0$ then $\frac{1}{n}(NM + \sum_{k=1}^{N} M_k) < \frac{\epsilon}{2}$. Hence for all $n \ge N_0$, and all $x \in E$, $\left| f(x) - \frac{f_1(x) + \cdots + f_n(x)}{n} \right| < \epsilon$.

Section 7.2, Exercise 1.

(a). To prove this, we use the Weierstrass M-test. Since $|\cos(t)| \leq 1$ for all $t \in \mathbf{R}$, then $|\cos(kx)/k^2| \leq 1/k^2$ for all $k \in \mathbf{N}$ and $x \in \mathbf{R}$. Since by the p-series test $\sum_k(1/k^2) < \infty$, the Weierstrass M-test implies that $\sum_k \cos(kx)/k^2$ converges absolutely and uniformly on $\mathbf{R}$.

(b). Using the well-known inequality $|\sin(t)| \leq |t|$ for all $t \in \mathbf{R}$, we have that $|\sin(x/k^2)| \leq |x|/k^2$ for all $x \in \mathbf{R}$ and $k \in \mathbf{N}$. Hence by the Comparison Test, if we show that $\sum_k(x/k^2)$ converges uniformly on any bounded interval, then it will follow that $\sum_k \sin(x/k^2)$ will do the same.

Let $[a, b] \subseteq \mathbf{R}$ be given and by choosing $R = \max(|a|, |b|)$ we can assert that $[a, b] \subseteq [-R, R]$. Then if $x \in [a, b]$, $|x| \leq R$ and $|x|/k^2 \leq R/k^2$. Since $\sum_k(R/k^2) = R\sum_k(1/k^2) < \infty$ since by the p-series test $\sum_k(1/k^2) < \infty$, the Weierstrass M-test implies that $\sum_k x/k^2$ and hence $\sum_k \sin(x/k^2)$ converges absolutely and uniformly on $\mathbf{R}$.

Exercise 2.

For this result we will apply the Weierstrass M-test. If $[a, b] \subseteq (-1, 1)$ then $a > -1$ and $b < 1$ so that choosing $r = \max(|a|, |b|)$ we have that $r < 1$ and that $[a, b] \subseteq [-r, r] \subseteq (-1, 1)$. If $x \in [a, b]$ then $|x| \leq r$ and $|x^k| = |x|^k \leq r^k$. Since $0 < r < 1$, $\sum_k r^k$ converges since it is a convergent geometric series. Therefore by the Weierstrass M-test the original series $\sum_k x^k$ converges absolutely and uniformly on $[a, b]$.