MATH 316 – HOMEWORK 2
SOLUTIONS TO SELECTED EXERCISES

Section 6.4, Exercise 3.

(a). Applying the Ratio Test we compute

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\left|\frac{(-1)^{k+1}(k+1)^3}{(k+2)!}\,\frac{(k+1)!}{(-1)^k k^3}\right| \\
&= \lim_{k\to\infty}\frac{(k+1)^3}{(k+2)!}\,\frac{(k+1)!}{k^3} \\
&= \lim_{k\to\infty}\left(\frac{k+1}{k}\right)^3\frac{1}{k+2} \\
&= 0
\end{aligned}$$

since $\lim\left(\frac{k+1}{k}\right)^3 = 1$ by L'Hopital's rule, and $\lim 1/(k+2) = 0$. Since $0 < 1$ the Ratio Test implies that the series converges absolutely.

(b). Applying the Ratio Test we compute

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\left|\frac{(-1)(-3)\cdots(1-2k)(-1-2k)}{(1)(4)\cdots(3k-2)(3k+1)}\,\frac{(1)(4)\cdots(3k-2)}{(-1)(-3)\cdots(1-2k)}\right| \\
&= \lim_{k\to\infty}\frac{(1)(3)\cdots(2k-1)(2k+1)}{(1)(4)\cdots(3k-2)(3k+1)}\,\frac{(1)(4)\cdots(3k-2)}{(1)(3)\cdots(2k-1)} \\
&= \lim_{k\to\infty}\frac{2k+1}{3k+1} \\
&= 2/3
\end{aligned}$$

by L'Hopital's rule. Since $2/3 < 1$ the Ratio Test implies that the series converges absolutely.

(c). Applying the Ratio Test we compute

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\left|\frac{(k+2)^{k+1}}{p^{k+1}(k+1)!}\,\frac{p^k\,k!}{(k+1)^k}\right| \\
&= \lim_{k\to\infty}\frac{1}{p}\,\frac{(k+2)^{k+1}k!}{(k+1)!(k+1)^k} \\
&= \lim_{k\to\infty}\frac{1}{p}\,\frac{(k+2)^{k+1}}{(k+1)(k+1)^k} \\
&= \lim_{k\to\infty}\frac{1}{p}\,\frac{(k+2)^{k+1}}{(k+1)^{k+1}} \\
&= \lim_{k\to\infty}\frac{1}{p}\left(\frac{k+2}{k+1}\right)^{k+1}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{k\to\infty} \frac{1}{p}\left(1+\frac{1}{k+1}\right)^{k+1} \\
&= \frac{e}{p}
\end{aligned}
$$

since $\lim(1+(1/k))^k = e$ by L'Hopital's rule. If $p > e$ then $e/p < 1$ and the Ratio Test implies that the series converges absolutely.

(d). Letting $a_n = \left|\frac{(-1)^{k+1}\sqrt{k}}{k+1}\right| = \frac{\sqrt{k}}{k+1}$ and $b_k = 1/\sqrt{k}$, L'Hopital's rule implies that $\lim_k a_k/b_k = k/(k+1) = 1$ and so by the Limit Comparison Test the series $\sum_k \sqrt{k}/(k+1)$ diverges with $\sum_k 1/\sqrt{k}$ the latter being a p-series with $p = 1/2 < 1$. Hence the original series does not converge absolutely.

On the other hand, the sequence $\sqrt{k}/(k+1)$ decreases to zero. To see this, let $f(x) = \sqrt{x}/(x+1)$. Then $f'(x) = (1-x)/(2\sqrt{x}(x+1)^2) < 0$ for $x > 1$. The Alternating Series Test implies that the original series converges and hence it converges conditionally.

(e). Applying the Root Test we compute

$$
\begin{aligned}
\lim_{k\to\infty} |a_k|^{1/k} &= \lim_{k\to\infty}\left(\frac{(k+1)^{1/2}}{k^{1/2}\,k^k}\right)^{1/k} \\
&= \lim_{k\to\infty}\left(\frac{k+1}{k}\right)^{1/2k}\frac{1}{k} \\
&= 0
\end{aligned}
$$

since $\lim[(k+1)/k]^{1/2k} = 1$ since it converges to the form 1^0 which is not indeterminate, and since $\lim 1/k = 0$. Since $0 < 1$ the Root Test implies that the series converges absolutely.

Exercise 4.

Since $\{b_n\}$ converges, it is bounded, and hence there is a number B such that for all k, $|b_k| \le B$. Since $\sum_k a_k$ converges, given $\epsilon > 0$ there is an N such that if $n \ge m \ge N$ then $|\sum_{k=m}^n| < \epsilon/(3B)$.

Now suppose that $n \ge m \ge N$. The summation by parts formula says that

$$
\sum_{k=m}^{n} a_k b_k = \left(\sum_{k=m}^{n} a_k\right) b_n - \sum_{k=m}^{n-1}\left(\sum_{j=m}^{k} a_j\right)(b_{k+1}-b_k).
$$

Since $\{b_k\}$ is decreasing we have that $|b_{k+1}-b_k| = (b_k - b_{k+1})$ and we compute

$$
\left|\sum_{k=m}^{n} a_k b_k\right| \le \left|\sum_{k=m}^{n} a_k\right||b_n| + \sum_{k=m}^{n-1}\left|\sum_{j=m}^{k} a_j\right||b_{k+1}-b_k|
$$

$$
\begin{aligned}
&\leq \; B\epsilon/(3B) + \sum_{k=m}^{n-1} \epsilon/(3B)(b_k - b_{k+1}) \\
&< \; B\,\epsilon/(3B) + \epsilon/(3B) \sum_{k=m}^{n-1} (b_k - b_{k+1}) \\
&= \; B\,\epsilon/(3B) + \epsilon/(3B)\,(b_m - b_n) \\
&\leq \; B\,\epsilon/(3B) + 2B\,\epsilon/(3B) = \epsilon.
\end{aligned}
$$

A slicker proof discovered by several of you is the following. Since $\sum_k a_k$ converges so does $\sum_k b\, a_k$. Also since the sequence of partial sums of $\sum_k a_k$ converges that sequence is bounded. Since b_k decreases to b, the sequence $(b_k - b)$ decreases to zero. Hence we can apply Dirichlet's Test and conclude that $\sum_k a_k(b_k - b)$ converges. Therefore

$$
\sum_k a_k\, b_k = \sum_k a_k(b_k - b) + \sum_k a_k b
$$

converges since the sum of two convergent series converges.

Exercise 8.

Let M be such that $|\sum_{k=m}^{n} b_k| \leq M$ for all $n \geq m \geq 1$. Such an M exists by the same argument used to prove Dirichlet's Test. By Summation by Parts, we have that

$$
\begin{aligned}
\left|\sum_{k=m}^{n} a_k\, b_k\right| \;&=\; \left|\left(\sum_{k=m}^{n} b_k\right) a_n - \sum_{k=m}^{n-1} \left(\sum_{j=m}^{k} b_j\right) (a_{k+1} - a_k)\right| \\
&\leq\; |a_n| \left|\sum_{k=m}^{n} b_k\right| + \sum_{k=m}^{n-1} \left|\sum_{j=m}^{k} b_j\right| |a_{k+1} - a_k| \\
&\leq\; M \left(|a_n| + \sum_{k=m}^{n-1} |a_{k+1} - a_k| \right)
\end{aligned}
$$

Given $\epsilon > 0$ choose N so large that if $n \geq m \geq N$ then $|a_n| < \epsilon/2M$ and $\sum_{k=m}^{n-1} |a_{k+1} - a_k| < \epsilon/2M$. The former inequality comes from the fact that $\lim a_n = 0$ and the latter from the fact that $\sum_{k=1}^{\infty} |a_{k+1} - a_k| < \infty$. Then for all such n and m,

$$
\left|\sum_{k=m}^{n} a_k\, b_k\right| \leq M \left(|a_n| + \sum_{k=m}^{n-1} |a_{k+1} - a_k| \right) < M(\epsilon/2M + \epsilon/2M) = \epsilon.
$$

Section 6.6, Exercise 1.

(a). If $a_k = 1/(\log(k))^{\log(k)}$ then $|a_k|^{1/k} = 1/(\log(k))^{\log(k)/k}$ and in order to evaluate $\lim_k |a_k|^{1/k}$ we need to use L'Hopital's Rule to evaluate $\lim_k (\log(k))^{\log(k)/k}$. Since this evaluates to the indeterminate form ∞^0 we take the logarithm and evaluate using L'Hopital's Rule,

$$\begin{aligned}
\lim_k \log[(\log(k))^{\log(k)/k}] &= \lim_k (\log(k)/k)\ \log(\log(k)) \\
&= \lim_k \frac{\log(k) \log(\log(k))}{k} \\
&= \lim_k \frac{\frac{1}{k} + \frac{1}{k}\ \log(\log(k))}{1} \\
&= \lim_k \frac{1}{k} + \lim_k \frac{\log(\log(k))}{k} \\
&= 0 + \lim_k \frac{\frac{1}{k\log(k)}}{1} \\
&= 0.
\end{aligned}$$

Hence, $\lim_k (\log(k))^{\log(k)/k} = e^0 = 1$ and also $\lim_k |a_k|^{1/k} = \lim_k 1/(\log(k))^{\log(k)/k} = 1$. Hence the Root Test fails.

Using the Logarithmic Test, we note that $\log(1/|a_k|) = \log[(\log(k))^{\log(k)}] = \log(k)\ \log(\log(k))$. Hence

$$\frac{\log(1/|a_k|)}{\log(k)} = \frac{\log(k)\ \log(\log(k))}{\log(k)} = \log(\log(k)) \to \infty$$

as $k \to \infty$. Since $\infty > 1$ the Logarithmic Test implies that the original series converges.

(b). Applying the Ratio Test we get

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{(1)(3)\cdots(2k-1)(2k+1)}{(4)(6)\cdots(2k+2)(2k+4)} \frac{(4)(6)\cdots(2k+2)}{(1)(3)\cdots(2k-1)} = \frac{2k+1}{2k+4} \to 1$$

as $k \to \infty$. Hence the Ratio Test fails.

However,

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{2k+1}{2k+4} = 1 - \frac{3}{2k+4} = 1 - \frac{3/2}{k+2}.$$

This satisfies Raabe's Test with $p = 3/2 > 1$ and we conclude that the series converges.