MATH 316 – HOMEWORK 1
SOLUTIONS TO SELECTED EXERCISES

Section 6.1, Exercise 5.

(a). $\lim_{k\to\infty} \cos\frac{1}{k^2} = 1$ so by Theorem 6.5, $\displaystyle\sum_{k=1}^{\infty} \cos\frac{1}{k^2}$ diverges.

(b). $\lim_{k\to\infty}\left(1-\frac{1}{k}\right)^k = e^{-1}$ (apply L'Hopital's rule) so by Theorem 6.5, $\displaystyle\sum_{k=1}^{\infty}\left(1-\frac{1}{k}\right)^k$ diverges.

(c). $\dfrac{k+1}{k^2} \geq \dfrac{k}{k^2} = \dfrac{1}{k}$. Therefore for each $n \in \mathbf{N}$, $\displaystyle\sum_{k=1}^{n}\frac{k+1}{k^2} \geq \sum_{k=1}^{n}\frac{1}{k}$. By Example 6.4, $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k} = \infty$ and hence $\displaystyle\sum_{k=1}^{n}\frac{k+1}{k^2}$ diverges.

Exercise 7.

(a). Since $nb = \displaystyle\sum_{k=1}^{n} b$, we can write

$$\begin{aligned}
\left|nb - \sum_{k=1}^{n} b_k\right| &= \left|\sum_{k=1}^{n} b - \sum_{k=1}^{n} b_k\right| \\
&= \left|\sum_{k=1}^{n}(b - b_k)\right| \\
&\leq \sum_{k=1}^{n}|b - b_k| \\
&= \sum_{k=1}^{N}|b - b_k| + \sum_{k=N+1}^{n}|b - b_k| \\
&\leq \sum_{k=1}^{N}|b - b_k| + M(n-(N+1)+1) \\
&= \sum_{k=1}^{N}|b - b_k| + M(n-N).
\end{aligned}$$

(b). Let $\epsilon > 0$ and choose N so large that if $k \geq N$ then $|b - b_k| < \epsilon/2$. Then for all $n > N$, applying part (a),

$$\begin{aligned}
\left|b - \frac{b_1 + \cdots + b_n}{n}\right| &= \frac{1}{n}\left|nb - \sum_{k=1}^{n} b_k\right| \\
&\leq \frac{1}{n}\sum_{k=1}^{N}|b - b_k| + \frac{\epsilon}{2}\frac{(n-N)}{n} \\
&\leq \frac{1}{n}\sum_{k=1}^{N}|b - b_k| + \frac{\epsilon}{2}
\end{aligned}$$

since $(n-N)/n < 1$ for all $n > N$.

Now choose N_0 so large that if $n \geq N_0$ then $\dfrac{1}{n}\sum_{k=1}^{N}|b-b_k| < \dfrac{\epsilon}{2}$. Hence for all $n \geq N_0$, $\left|b - \dfrac{b_1+\cdots+b_n}{n}\right| < \epsilon$.

(c). If $b_k = (-1)^{k+1}(1/k)$, then $b_1 + b_2 + \cdots + b_n = 1$ if n is odd and 0 if n is even. Therefore, $\left|\dfrac{b_1+\cdots+b_n}{n}\right| = \dfrac{1}{n}$ if n is odd and 0 if n is even. Hence $\lim_{n\to\infty}\left|\dfrac{b_1+\cdots+b_n}{n}\right| = 0$ but $\lim_{k\to\infty} b_k \neq 0$ since the limit does not exist. Hence the converse of (b) is false.

Exercise 9.

(a). We will prove first that $\lim_{n\to\infty}(2n)\,a_{2n} = \lim_{n\to\infty}(2n+1)\,a_{2n+1} = 0$ from which it will follow immediately that $\lim_{n\to\infty} n\,a_n = 0$ (This follows from the definition of convergence of a sequence. I will leave the details to you.)

Since $\sum_{k=1}^{\infty} a_k$ converges, its partial sums satisfy a Cauchy criterion which implies that $\lim_{n\to\infty}\sum_{k=n+1}^{2n} a_k = \lim_{n\to\infty}\sum_{k=n+1}^{2n+1} a_k = 0$. (I will again leave the verification of this fact to you.) Now

$$\sum_{k=n+1}^{2n} a_k = a_{n+1} + a_{n+2} + \cdots + a_{2n} \geq a_{2n} + a_{2n} + \cdots + a_{2n} = na_{2n}$$

since $a_{2n} \leq a_k$ for all $k \geq 2n$ since $\{a_k\}$ is a decreasing sequence. Similarly,

$$\sum_{k=n+1}^{2n+1} a_k = a_{n+1}+a_{n+2}+\cdots+a_{2n+1} \geq a_{2n+1}+a_{2n+1}+\cdots+a_{2n+1} = (n+1)a_{2n+1}.$$

Therefore, since $(2n)a_{2n} \leq 2\sum_{k=n+1}^{2n} a_k$, the Squeeze Theorem (Theorem 2.9) implies that $\lim_{n\to\infty}(2n)\,a_{2n} = 0$. Similarly, since $(2n+1)\,a_{2n+1} \leq (2n+2)\,a_{2n+1} \leq \sum_{k=n+1}^{2n+1} a_k$, the Squeeze Theorem implies that $\lim_{n\to\infty}(2n+1)\,a_{2n+1} = 0$. Hence $\lim_{k\to\infty} k\,a_k = 0$.

(b). If we let $b_n = s_{2n}$, then

$$\begin{aligned} b_{n+1} - b_n &= s_{2n+2} - s_{2n} = (-1)^{2n+3}(1/(2n+2)) + (-1)^{2n+2}(1/(2n+1)) \\ &= 1/(2n+1) - 1/(2n+2) > 0. \end{aligned}$$

Therefore $b_{n+1} > b_n$ and $\{b_n\} = \{s_{2n}\}$ is strictly increasing. Similarly, if $c_n = s_{2n+1}$, then

$$\begin{aligned} c_{n+1} - c_n &= s_{2n+3} - s_{2n+1} = (-1)^{2n+4}(1/(2n+3)) + (-1)^{2n+1}(1/(2n+2)) \\ &= 1/(2n+3) - 1/(2n+2) < 0. \end{aligned}$$

Therefore $c_{n+1} < c_n$ and $\{c_n\} = \{s_{2n+1}\}$ is strictly decreasing. Finally note that $s_{2n+1} - s_{2n} = (-1)^{2n+2}(1/(2n+1)) = 1/(2n+1) \to 0$ as $n \to \infty$.

(c). Note that in the last part of part (b) we have shown that $s_{2n+1} \geq s_{2n}$ for each n. This means that the sequence $\{s_{2n+1}\}$ is decreasing and bounded below and that the sequence $\{s_{2n}\}$ is increasing and bounded above. Hence each is convergent (Theorem 2.19). Since $\lim_{n\to\infty} s_{2n+1} - s_{2n} = 0$, the two limits must be the same. Moreover, $\lim_{n\to\infty} s_n$ exists and equals this same number. Hence $\sum_{k=1}^{\infty}(-1)^{k+1}(1/k)$ converges. But letting $a_k = (-1)^{k+1}(1/k)$ we have that $\sum a_k$ converges, but $k\,a_k = (-1)^{k+1}$, and $\lim_{k\to\infty} k\,a_k$ does not exist. Therefore part (a) is false if the sequence $\{a_k\}$ of terms is not decreasing.

Section 6.2, Exercise 1.

(a). Use the Limit Comparison test with the series $\sum 1/k^2$.

(b). Use Limit Comparison or Direct Comparison test with $\sum 1/2^k$.

(c). Use the Integral Test (You need to verify that the sequence of terms is decreasing. The integral can be evaluated by parts.)

(d). Use Limit Comparison or Direct Comparison test with $\sum 1/3^k$.

(e). Use Limit Comparison test with $\sum 10\,k^{-e}$ and observe that $e > 1$.

(f). Use the Limit Comparison test with the series $\sum 1/k^2$.

Exercise 2.

(a). Use the Limit Comparison test with the series $\sum 1/k$.

(b). Use the Integral Test or Direct Comparison with $\sum 1/k$.

(c). Use Direct Comparison with $\sum 1/k$.

(d). Use the Integral Test, evaluating the integral by the substitution $u = \log^p(x)$.

Exercise 10.

Suppose that $\sum_k a_k$ converges. This means that for every $\epsilon > 0$, there exists N such that if $m \geq n \geq N$ then $|\sum_{k=n}^{m} a_k| < \epsilon$. Now choose $M \geq N/2$ and suppose that $m \geq n \geq M$. Then

$$\left|\sum_{k=n}^{m}(a_{2k} + a_{2k+1})\right| = \left|\sum_{k=2n}^{2m+1} a_k\right| < \epsilon$$

since $n \geq M \geq N/2$ implies that $2n \geq N$ and $m \geq n$ implies $2m+1 \geq 2n \geq N$. Hence the series $\sum_{k=1}^{\infty}(a_{2k} + a_{2k+1})$ satisfies a Cauchy condition hence converges.

Suppose now that $\sum_k (a_{2k} + a_{2k+1})$ converges. This means that for every $\epsilon > 0$, there exists N so that if $m \geq n \geq N_0$ then $|\sum_{k=n}^{m}(a_{2k} + a_{2k+1})| = |\sum_{k=2n}^{2m+1} a_k| < \epsilon/3$ and $a_k \to 0$ as $k \to \infty$ implies that there is an N_1 such that if $k \geq N_1$ then $|a_k| < \epsilon/3$. Choose $N \geq \max(N_0, N_1)$. Given $m \geq n \geq 2N$, if $m = 2k + 2$ is even and $n = 2j + 1$ is odd then $j = (n-1)/2 \geq (2N-1)/2 \geq N$ and similarly $k \geq N$, and

$$\left|\sum_{k=n}^{m} a_k\right| = \left|a_n + \sum_{k=2j}^{2k+1} a_k + a_m\right| \leq |a_n| + \left|\sum_{k=2j}^{2k+1} a_k\right| + |a_m|.$$

If m is odd or n is even then the same identity holds without one or both of the a_n or a_m terms. In any case, then,

$$\left|\sum_{k=n}^{m} a_k\right| \leq |a_n| + \left|\sum_{k=2j}^{2k+1} a_k\right| + |a_m| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

Section 6.3, Exercise 2.

(a). Use the Root Test, concluding that $\lim_{k\to\infty}(k^2/\pi^k)^{1/k} = \lim_{k\to\infty} k^{2/k}/\pi = 1/\pi < 1$ and hence that the series converges.

(b). Use the Ratio Test and conclude that the series diverges.

(c). Use the Root Test, concluding that $\lim_{k\to\infty}\left[\left(\frac{k+1}{2k+3}\right)^k\right]^{1/k} = \lim_{k\to\infty}\left(\frac{k+1}{2k+3}\right) = 1/2 < 1$ and hence that the series converges.

(d). Use the Limit Comparison Test with $\sum_k \pi/k$ and conclude that the series diverges.

Exercise 6.

(a). Use the Integral Test to show series converges if and only if $p > 1$.

(b). Use the Limit Comparison Test to compare the series with $\sum_k 1/k$ and conclude that the series diverges for all $p \in \mathbf{R}$.

(c). Use the Root Test or Ratio Test to conclude that the series converges if and only if $|p| > 1$.