MATH 316 – SPRING 2009 – MIDTERM EXAM – SOLUTIONS

1. (5 pts. each) Determine whether each of the following series converges. All calculations must be fully justified and any conclusions you reach must be justified by the correct application of an appropriate convergence test.

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{k^{2k}}{(3k^2+k)^k}$.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(3)(6)\cdots(3k)}{(7)(10)\cdots(3k+4)}$. (Hint: The Ratio Test does not work.)

Solution:

(a). Use the Root Test. Let $a_k = \dfrac{k^{2k}}{(3k^2+k)^k}$. Then

$$|a_k|^{1/k} = \left(\frac{k^{2k}}{(3k^2+k)^k}\right)^{1/k} = \frac{k^2}{3k^2+k} \to \frac{1}{3}$$

as $k \to \infty$ by L'Hopital's Rule. Therefore, $\limsup_{k\to\infty} |a_k|^{1/k} = 1/3 < 1$ and by the Root test the series converges.

(b). Use Raabe's Test. Let $a_k = \dfrac{(3)(6)\cdots(3k)}{(7)(10)\cdots(3k+4)}$. Then

$$\begin{aligned}\frac{a_{k+1}}{a_k} &= \frac{(3)(6)\cdots(3k)(3k+3)}{(7)(10)\cdots(3k+4)(3k+7)} \cdot \frac{(7)(10)\cdots(3k+4)}{(3)(6)\cdots(3k)} \\ &= \frac{3k+3}{3k+7} = 1 - \frac{4}{3k+7} = 1 - \frac{(4/3)}{k+(7/3)}.\end{aligned}$$

Since $4/3 > 1$, Raabe's Test implies that the series converges.

2. (10 pts.) Prove that the sequence of functions $\{x^n\}_{n=1}^{\infty}$ converges *pointwise* but not *uniformly* to 0 on the interval $(0,1)$.

Solution:

Let $x_0 \in (0,1)$. Since then $|x_0| < 1$, $\lim_{n\to\infty} x_0^n = 0$. Hence $x^n \to 0$ as $n \to \infty$ pointwise on $(0,1)$.

To see that the convergence is not uniform note that $x^n \to 0$ on $(0,1)$ uniformly means that $\sup_{x\in(0,1)} |x^n| \to 0$ as $n \to \infty$. But in fact, since $\lim_{x\to 1^-} x^n = 1$ for all $n \in \mathbf{N}$, it follows that for all $n \in \mathbf{N}$, $\sup_{x\in(0,1)} |x^n| = 1$ and so it does not go to zero.

Another way to approach this is the following: $x^n \to 0$ on $(0,1)$ uniformly means that for all $\epsilon > 0$ there is an N so that if $n \geq N$ then $|x^n| < \epsilon$ for all $x \in (0,1)$. However, if $\epsilon = 1/2$ then since $\lim_{x\to 1^-} x^n = 1$ for all $n \in \mathbf{N}$, then for any $n \in \mathbf{N}$, we can find an $x \in (0,1)$ such that $x^n > 1/2$ (of course this x will depend on n but that does not matter). Hence the convergence is not uniform.

3. (5 pts. each) Consider the series $\sum_{k=1}^{\infty} \frac{\cos(kx)}{2^k}$.

(a) Show that the above series converges uniformly on $\mathbf{R}$ to a continuous function $f(x)$.

(b) Show that the function $f(x)$ found in part (a) is continuously differentiable on $\mathbf{R}$ and that $f'(x)$ is bounded on $\mathbf{R}$.

Solution:

(a). Since $|\cos(kx)| \le 1$ for all $k \in \mathbf{N}$ and $x \in \mathbf{R}$, letting $M_k = 1/2^k$ it follows that for all $k \in \mathbf{N}$ and $x \in \mathbf{R}$, $|\cos(kx)/2^k| \le M_k$, and since $\sum_k 1/2^k$ is a convergent geometric series, that $\sum_k M_k < \infty$. By the Weierstrass M-test, the series $\sum_k \cos(kx)/2^k$ converges absolutely and uniformly on $\mathbf{R}$. To see that the limit function is continuous, note that for each k, $\cos(kx)/2^k$ is continuous on $\mathbf{R}$ and since the convergence is uniform, the series converges to a continuous function.

(b). In order to apply the term-by-term differentiation test, we must establish three things: (1) that for each k, $\cos(kx)/2^k$ is differentiable, (2) that the series $\sum_k \cos(kx)/2^k$ converges at a point in $\mathbf{R}$, and (3) that the series of derivatives, $\sum_k d/dx[\cos(kx)/2^k]$ converges uniformly on $\mathbf{R}$. To see (1), note that $d/dx[\cos(kx)/2^k] = (-k\sin(kx))/2^k$ by the usual rules of differentiation. To see (2), note that by part (a) the series $\sum_k cos(kx)/2^k$ converges at each point of $\mathbf{R}$ hence at a single point. To see (3), note that for each $k \in \mathbf{N}$ and $x \in \mathbf{R}$, $|\sin(kx)| \le 1$, so that letting $M_k = k/2^k$ it follows that for all $k \in \mathbf{N}$ and $x \in \mathbf{R}$, $|k\sin(kx)/2^k| \le M_k$. Since the series $\sum_k M_k = \sum_k (k/2^k)$ converges by the Ratio Test ($|a_{k+1}/a_k| = (k+1)/2^{k+1} \cdot 2^k/k = [(k+1)/k](1/2) \to 1/2 < 1$ as $k \to \infty$ by L'Hoptial's Rule) the Weierstrass M-test implies that the series $\sum_k d/dx[\cos(kx)/2^k] = -\sum_k k\sin(kx)/2^k$ converges absolutely and uniformly on $\mathbf{R}$. Hence the function $f(x) = \sum_k \cos(kx)/2^k$ is differentiable term-by-term and $f'(x) = -\sum_k k\sin(kx)/2^k$.

To see that in fact $f'(x)$ is continuous, note that the series converges uniformly on $\mathbf{R}$ and that each term $k\sin(kx)/2^k$ is continuous on $\mathbf{R}$. Hence so is $f'(x)$. To see that $f'(x)$ is bounded, note that $|f'(x)| = |\sum_k k\sin(kx)/2^k| \le \sum_k k|\sin(kx)|/2^k \le \sum_k k/2^k < \infty$.

4. (10 pts.) Find the interval of convergence of the power series $\sum_{k=0}^{\infty} [(-1)^k + 2]^k x^k$.

Solution:

To find the radius of convergence we compute

$$\limsup_{k\to\infty} |([(-1)^k + 2]^k)|^{1/k} = \limsup_{k\to\infty} |(-1)^k + 2| = \limsup_{k\to\infty}((-1)^k + 2).$$

Now, $((-1)^k + 2) = 1$ if k is odd and 3 if k is even, so $\limsup_{k\to\infty}((-1)^k + 2) = 3$ and the radius of convergence is $1/3$. Hence the power series converges absolutely on $(-1/3, 1/3)$ and diverges outside $[-1/3, 1/3]$. What remains is to check convergence at $x = -1/3$ and $x = 1/3$. When $x = 1/3$,

$$[(-1)^k + 2]^k x^k = [(-1)^k + 2]^k (\frac{1}{3})^k = \left(\frac{(-1)^k + 2}{3}\right)^k.$$

This term equals $1/3^k$ if k is odd and 1 if k is even. Hence the sequence of terms does not converge to zero and the series diverges. When $x = -1/3$,

$$[(-1)^k + 2]^k x^k = [(-1)^k + 2]^k (-\frac{1}{3})^k = \left(\frac{(-1)^{k+1} - 2}{3}\right)^k.$$

This term equals $(-1/3)^k$ if k is odd and $(-1)^k$ if k is even. Hence the sequence of terms does not converge to zero and the series diverges. Therefore the interval of convergence is $(-1/3, 1/3)$.

5. (10 pts.) Assume that $\sum_{k=0}^{\infty}(-1)^k \dfrac{x^{2k+1}}{2k+1} = \arctan(x)$ for $x \in (-1, 1)$. Prove that

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}.$$

Each step in your proof must be fully justified as in the instructions to this exam.

Solution:

When $x = 1$, $\sum_{k=0}^{\infty}(-1)^k \dfrac{x^{2k+1}}{2k+1} = \sum_{k=0}^{\infty} \dfrac{(-1)^k}{2k+1}$. Since the sequence $1/(2k+1)$ decreases to 0 as $k \to \infty$, the Alternating Series Test implies that the series $\frac{(-1)^k}{2k+1}$ converges.

By Abel's Theorem, since the power series $\sum_{k=0}^{\infty}(-1)^k \dfrac{x^{2k+1}}{2k+1}$ converges at $x = 0$ and at $x = 1$ (the latter by the Alternating Series Test, see above), the series converges uniformly on the interval $[0, 1]$. Since each term in the series is continuous on $[0, 1]$ (being a multiple of a monomial), the limit function is continuous on $[0, 1]$. Hence the left-hand limit at $x = 1$ of this function can be computed by simply evaluating the function at $x = 1$. That is,

$$\lim_{x\to 1^-} \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}.$$

But since $\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1} = \arctan(x)$ on $[0, 1)$, $\arctan(1) = \pi/4$, and since $\arctan(x)$ is continuous on $\mathbf{R}$,

$$\frac{\pi}{4} = \arctan(1) = \lim_{x \to 1^-} \arctan(x) = \lim_{x \to 1^-} \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

as desired.