MATH 316 – SPRING 2009 – MIDTERM EXAM

Answer all of the following questions on the sheets provided. You are to show all work but try to be as neat as possible. It is required only that your solutions be legible and reasonably well organized. You may turn in your scratch work if you like but make sure that it is clear to me what is your final solution and what is scratch work. All steps in your proofs must be justified either by an appropriate calculation or the citation of an appropriate theorem. In each case where a theorem is cited, it must be shown that all hypotheses of the theorem are satisfied.

1. (5 pts. each) Determine whether each of the following series converges. All calculations must be fully justified and any conclusions you reach must be justified by the correct application of an appropriate convergence test.

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{k^{2k}}{(3k^2+k)^k}$.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(3)(6)\cdots(3k)}{(7)(10)\cdots(3k+4)}$. (Hint: The Ratio Test does not work.)

2. (10 pts.) Prove that the sequence of functions $\{x^n\}_{n=1}^{\infty}$ converges *pointwise* but not *uniformly* to 0 on the interval $(0,1)$.

3. (5 pts. each) Consider the series $\displaystyle\sum_{k=1}^{\infty} \frac{\cos(kx)}{2^k}$.

(a) Show that the above series converges uniformly on $\mathbf{R}$ to a continuous function $f(x)$.

(b) Show that the function $f(x)$ found in part (a) is continuously differentiable on $\mathbf{R}$ and that $f'(x)$ is bounded on $\mathbf{R}$.

4. (10 pts.) Find the interval of convergence of the power series $\displaystyle\sum_{k=0}^{\infty} [(-1)^k + 2]^k x^k$.

5. (10 pts.) Assume that $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = \arctan(x)$ for $x \in (-1,1)$. Prove that $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}$. Each step in your proof must be fully justified as in the instructions to this exam.