Example Fourier series

A Fourier series is a series of the form

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx).$$

Q: For which x does a given F.S. converge? If converges for all x then the resulting function is 2π-periodic, i.e. $f(x+2\pi) = f(x)$ all x.

Q: Is every 2π-periodic function equal to a Fourier series? If so, is that Fourier series unique?

Thm: Suppose $a_k \downarrow 0$. Then $\sum_{k=1}^{\infty} a_k \cos(kx)$ converges for every $x \in (0, 2\pi)$.

~~Pf.~~ Idea: Use Dirichlet test, so must examine $s_n = \sum_{k=1}^{n} \cos(kx)$. Is the sequence bounded? YES!

Why? Recall that $e^{ix} = \cos x + i \sin x$ so $e^{ikx} = \cos(kx) + i\sin(kx)$. Using sum formula for geometric series, $\sum_{k=1}^{n} r^k = \frac{r - r^{n+1}}{1-r}$ gives

$$\sum_{k=1}^{n} e^{ikx} = \sum_{k=1}^{n} (e^{ix})^k = \frac{e^{ix} - e^{i(n+1)x}}{1 - e^{ix}} \cdot \frac{e^{-ix/2}}{e^{-ix/2}}$$

$$= \frac{e^{ix/2} - e^{i(n-1/2)x}}{e^{-ix/2} - e^{ix/2}}$$

$$\cos kx = \frac{1}{2}\left(e^{ikx} + e^{-ikx}\right)$$

$$\sum_{k=1}^{n} \cos kx = \frac{1}{2} \sum_{k=1}^{n}\left(e^{ikx} + e^{-ikx}\right)$$

$$\cancel{= \frac{1}{2} \sum_{k=-n}^{n} e^{ikx}}$$

$$= \frac{e^{ix/2} - e^{i(n-1/2)x/2}}{e^{-ix/2} - e^{ix/2}} + \frac{e^{-ix/2} - e^{-i(n-1/2)x/2}}{e^{ix/2} - e^{-ix/2}}$$

$$= \frac{\left(e^{-ix/2} - e^{ix/2}\right) - \left(e^{-i(n-1/2)x/2} - e^{i(n-1/2)x/2}\right)}{e^{ix/2} - e^{-ix/2}}$$

$$= \frac{-\cancel{2} \sin\frac{x}{2} + \cancel{2} \sin(n-1/2)\frac{x}{2}}{\cancel{2} \sin\frac{x}{2}}$$

$$\therefore \left|\sum_{k=1}^{n} \cos kx\right| \le \frac{2}{\left|\sin\frac{x}{2}\right|} \quad \text{if } x \in (0, 2\pi), \ \sin\frac{x}{2} \neq 0$$

<u>Lemma:</u> If $x \in (0, 2\pi)$ then $S_n = \sum_{k=1}^{n} \cos(kx)$ is a bounded sequence.

Thm 6.4 (Raabe Test)

Given $\{a_k\} \subseteq \mathbb{R}$ suppose that for some $p \in \mathbb{R}$,

$\left|\frac{a_{k+1}}{a_k}\right| \le 1 - \frac{p}{k}$ for k large. If $p > 1$, $\sum |a_k|$ converges.

Rem: 1) Note that here $\limsup_k \left|\frac{a_{k+1}}{a_k}\right| \le 1$, not < 1, so Ratio test cannot be used. Point of Raabe is that if $\left|\frac{a_{n+1}}{a_n}\right| \to 1$ but does so slowly enough, $\sum |a_k|$ still converges.

2) $\left|\frac{a_{k+1}}{a_k}\right| \le 1 - \frac{p}{k} < 1$ so $|a_{k+1}| < |a_k|$ all k large.

Thus $|a_k|$ decreases presumably to zero. Proof of Raabe relies on comparing this decrease to that of $\frac{1}{k^p}$ and showing $|a_k| \downarrow 0$ faster than $\frac{1}{k^p}$.

3) Let $x_k = \frac{1}{(k-1)^p}$. Then $\left|\frac{x_{k+1}}{x_k}\right| = \frac{(k-1)^p}{k^p} = \left(1 - \frac{1}{k}\right)^p$

Compare $\left(1-\frac{1}{k}\right)^p$ with $\left(1 - \frac{p}{k}\right)$. If $p \in \mathbb{N}$ then binomial formula gives

$$\left(1-\frac{1}{k}\right)^p = \sum_{j=0}^{p} \binom{p}{j}\left(\frac{-1}{k}\right)^j = 1 - \frac{p}{k} + \binom{p}{2}\frac{1}{k^2} - \binom{p}{3}\frac{1}{k^3} + \cdots + (-1)^p \frac{1}{k^p}$$

$$= \left(1 - \frac{p}{k}\right) + \frac{1}{k^2}\left[\binom{p}{2} + \text{stuff}\right]$$ where stuff $\to 0$ as $k \to \infty$.

So $\left(1-\frac{p}{k}\right) < \left(1-\frac{1}{k}\right)^p$ for k large. In fact we have such an inequality for any $p \geq 1$.

Lemma (Bernoulli inequality, p 97)

If $\delta \geq -1$ and $\alpha \geq 1$, $1+\alpha\delta \leq (1+\delta)^\alpha$

Pf (Uses MVT). Let $f(x) = x^\alpha$. Then $f'(x) = \alpha x^{\alpha-1}$ and by MVT, $f(1+\delta) = f(1) + f'(c)\delta = f(1) + \alpha c^{\alpha-1}\delta$ for some c between 1 and $1+\delta$. If $\delta > 0$ then $c > 1$ and since $\alpha \geq 1$, $\alpha - 1 \geq 0$. Thus $c^{\alpha-1} \geq 1$ and $f(1) + \alpha c^{\alpha-1}\delta \geq f(1) + \alpha\delta$. If $-1 \leq \delta \leq 0$ then $c \leq 1$ and $c^{\alpha-1} \leq 1$ thus $\alpha c^{\alpha-1} \leq \alpha$ and $\alpha c^{\alpha-1}\delta \geq \alpha\delta$ so again $f(1) + \alpha c^{\alpha-1}\delta \geq f(1) + \alpha\delta$.

Finally we have

$$(1+\delta)^\alpha = f(1+\alpha) \geq f(1) + \alpha\delta = 1 + \alpha\delta.$$

For our purposes we have that $\left(1-\frac{p}{k}\right) \leq \left(1-\frac{1}{k}\right)^p$ for all $p \geq 1$ and k large.

Point is that this shows $x_k \downarrow 0$ slower than $|a_k| \downarrow 0$. Proof of Raabe makes this explicit.

Pf: Let $x_k = \frac{1}{(k-1)^p}$ for $p \geq 1$ and consider the sequence $\left\{ \frac{|a_k|}{x_k} \right\} = \{b_k\}$ Now

$$\frac{b_{k+1}}{b_k} = \frac{|a_{k+1}|\, x_k}{|a_k|\, x_{k+1}} = \left|\frac{a_{k+1}}{a_k}\right| \frac{x_k}{x_{k+1}} \leq \left(1-\frac{p}{k}\right)\left(\frac{k}{k-1}\right)^p$$

$$= \left(1-\frac{p}{k}\right)\left(1-\frac{1}{k}\right)^{-p} \leq 1 \quad \text{by Bernoulli.}$$

Hence $b_{k+1} \leq b_k$ and $\{b_k\}$ decreases, and hence is convergent. By limit comparison test $\sum |a_k|$ converges if $\sum x_k$ does. But if $p > 1$, $\sum x_k$ converges.

Rem: replacing $1 - \frac{p}{k}$ by $1 - \frac{p}{k+c}$ has no effect on the proof if we replace $x_k = \frac{1}{(k-1)^p}$ with $x_k = \frac{1}{(k+c-1)^p}$ as (essentially) the book does.

Example: Hypergeometric series.

Let $\alpha, \beta > 0$ and consider

$$\sum_{k=0}^{\infty} \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k)}{\beta(\beta+1)(\beta+2)\cdots(\beta+k)} = \frac{\alpha}{\beta} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)} + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)} + \cdots$$

Applying Ratio test:

$$\frac{a_{k+1}}{a_k} = \frac{\alpha(\alpha+1)\cdots(\alpha+k)(\alpha+k+1)}{\beta(\beta+1)\cdots(\beta+k)(\beta+k+1)} \cdot \frac{\beta(\beta+1)\cdots(\beta+k)}{\alpha(\alpha+1)\cdots(\alpha+k)} = \frac{\alpha+k+1}{\beta+k+1}$$

$$\to 1 \text{ as } k \to \infty.$$

However $$\frac{\alpha+k+1}{\beta+k+1} - 1 = \frac{\alpha+k+1-\beta-k-1}{\beta+k+1} = \frac{\alpha-\beta}{\beta+k+1}$$

so that $$\left|\frac{a_{k+1}}{a_k}\right| = 1 - \frac{\beta-\alpha}{k+(\beta+1)}$$

By Raabe, if $\beta-\alpha>1$ series converges.

If $\beta-\alpha \le 1$ then $\beta \le 1+\alpha$ and

$$\frac{\alpha(\alpha+1)\cdots(\alpha+k)}{\beta(\beta+1)\cdots(\beta+k)} \ge \frac{\alpha(\alpha+1)\cdots(\alpha+k)}{(\alpha+1)(\alpha+2)\cdots(\alpha+k+1)} = \frac{\alpha}{\alpha+k+1}$$

So series diverges using comparison test with $\sum_{k=0}^{\infty} \frac{\alpha}{\alpha+k+1}$ which behaves like $\sum \frac{1}{k}$.