16.1. Laurent Series.

A. Laurent's Theorem.

1. *Theorem 1.* Let $f(z)$ be analytic an a domain D containing an annulus $\{z\colon \rho_1 < |z - z_0| < \rho_2\}$ and its boundary curves C_1 (outer) and C_2 (inner). Then we can write

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

   where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^{n+1}}\, dw$$

   and

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^{-n+1}}\, dw = \frac{1}{2\pi i} \int_C f(w)(z - z_0)^{n-1}\, dw$$

   and where C is a contour in the annulus enclosing C_2.

2. The series

$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

   is called the *principal part* of $f(z)$ at z_0.

3. The Laurent series converges in the largest annulus centered at z_0 on which $f(z)$ is analytic. Often we choose z_0 to be such that $f(z)$ is analytic except at $z = z_0$ so that the annulus of convergence has the form $\{z\colon 0 < |z - z_0| < R\}$.

4. Sometimes the principal part of $f(z)$ is zero, that is, $b_n = 0$ for $n = 1, 2, \ldots$, or is only a finite sum, that is, $b_n = 0$ for $n = N + 1, N + 2, \ldots$.

B. Uniqueness for Laurent Series.

1. A function $f(z)$ can have different Laurent series for different non-overlapping annuli with the same center.

2. If $f(z)$ has a singularity at $z = z_0$ then the *Laurent series near the singularity* z_0 is the Laurent series convergent in the annulus $\{z\colon 0 < |z - z_0| < R\}$.