15.1. Sequences, Series and Convergence.

A. Series of Complex Numbers.

1. A *series* $\sum_{m=1}^{\infty} z_m$ *converges to* $L \in \mathbf{C}$ if

$$\lim_{n\to\infty} \sum_{m=1}^{n} z_m = L,$$

that is, if

$$\lim_{n\to\infty} \left| \sum_{m=1}^{n} z_m - L \right| = 0.$$

2. By defining the *sequence of partial sums* by

$$s_n = \sum_{m=1}^{n} z_m,$$

convergence of the series $\sum_{m=1}^{\infty} z_m$ is the same as convergence of the *sequence* s_n.

3. If $z_m = x_m + iy_m$, then

$$\sum_{m=1}^{\infty} z_m = \sum_{m=1}^{\infty} x_m + i \sum_{m=1}^{\infty} y_m$$

converges if and only if $\sum_{m=1}^{\infty} x_m$ and $\sum_{m=1}^{\infty} y_m$ both converge.

B. Some Theorems on Convergence.

1. *Theorem 1.* If $\sum_{m=1}^{\infty} z_m$ converges then $\lim_{m\to\infty} z_m = 0$, but not the other way around.

2. *Cauchy's Criterion.* $\sum_{m=1}^{\infty} z_m$ converges if and only if

$$\left| \sum_{m=k}^{n} z_m \right| \to 0$$

as k, $n \to \infty$.

3. *Absolute Convergence.* If $\sum_{m=1}^{\infty} |z_m|$ converges, that is, if $\sum_{m=1}^{\infty} z_m$ *converges absolutely*, then $\sum_{m=1}^{\infty} z_m$ converges, but not the other way around.

C. Geometric Series.

1. A *geometric series* has the form $\sum_{m=0}^{\infty} q^m$ for some $q \in \mathbf{C}$.

2. The partial sums of a geometric series are given by

$$s_n = \sum_{m=0}^{n} q^m = \frac{1 - q^{n+1}}{1 - q}$$

if $q \neq 1$.

3. *Theorem 2.* $\sum_{m=0}^{\infty} q^m$ converges to $\frac{1}{1-q}$ if $|q| < 1$ and *diverges* (that is, does not converge) if $|q| \geq 1$.

D. The Ratio and Root Tests.

1. Suppose that $\sum_{m=1}^{\infty} z_m$ satisfies

$$\lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right| = L.$$

Then

(a) If $L < 1$ then $\sum z_m$ converges absolutely.
(b) If $L > 1$ then $\sum z_m$ diverges.
(b) If $L = 1$ then convergence of $\sum z_m$ cannot be determined by this test.

2. The idea of the ratio test is that if $z_m = q^m$ then $\sum_{m=1}^{\infty} z_m$ is a geometric series and in this case $|z_{n+1}/z_n| = |q|$ and convergence depends on whether or not $|q| < 1$ or $|q| \geq 1$. If $\lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right| = L$, then z_m *behaves like* L^m for large m so that $\sum z_m$ behaves like $\sum L^m$.

3. Suppose that $\sum_{m=1}^{\infty} z_m$ satisfies

$$\lim_{n\to\infty} (|z_n|)^{1/n} = L.$$

Then

(a) If $L < 1$ then $\sum z_m$ converges absolutely.
(b) If $L > 1$ then $\sum z_m$ diverges.
(b) If $L = 1$ then convergence of $\sum z_m$ cannot be determined by this test.

4. If $z_m = q^m$ then $\sum_{m=1}^{\infty} z_m$ is a geometric series and in this case $(|z_n|)^{1/n} = |q|$ and convergence depends on whether or not $|q| < 1$ or $|q| \geq 1$. If $\lim_{n\to\infty}(|z_n|)^{1/n} = L$, then z_m *behaves like* L^m for large m so that $\sum z_m$ behaves like $\sum L^m$.