14.2. Cauchy's Integral Theorem.

A. Contours.

1. A *simple closed path* in **C** is a curve $z(t)$ that does not touch itself. Such a curve is sometimes called a *contour*. That is, $z(t) = x(t) + iy(t)$.

2. A domain D is *simply connected* if every simple closed path in D encloses only points in D. That is, D has no holes.

B. Cauchy's Integral Theorem.

1. *Theorem 1.* If $f(z)$ is analytic in a simply connected domain D and if C is a contour in D then $\int_C f(z)\,dz = 0$.

2. Note that Cauchy's integral theorem is like independence of path, only with a twist.

$$\begin{aligned}\int_C f(z)\,dz &= \int_C (u+iv)(dx+idy)\\ &= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx - u\,dy)\\ &= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\,dA + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\,dA = 0\end{aligned}$$

   by the C-R equations and Green's Theorem.

C. Deformation of Path.

1. *Theorem 2.* If $f(z)$ is analytic in a simply connected domain D then the integral of f is independent of path provided that the paths are all contained in D.

2. If the endpoints of a path C are fixed, and if we can continuously deform C to another path C' with the same endpoints, then

$$\int_C f(z)\,dz = \int_{C'} f(z)\,dz$$

   as long as all intermediate paths between C and C' contain only points where $f(z)$ is analytic.

3. Given $f(z)$ analytic in D and some $z_0 \in D$, we can define for $z \in D$,

$$F(z) = \int_{z_0}^{z} f(z')dz'$$

   where the integral is taken over any path from z_0 to z that is contained in D. Then $F'(z) = f(z)$ and in particular, $F(z)$ is analytic in D.

4. If D is a doubly connected domain with boundary curves C_1 and C_2 and if f is analytic in a domain containing D and its boundary, then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$