10.2. Path Independence of Line Integrals.

A. Path Independence.

1. An integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is *path independent* if it has the same value for all curves C with the same endpoints. That is, its value depends only on the endpoints of C, not on C itself.

2. $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent in a domain D if and only if $\mathbf{F} = \nabla f$ for some scalar field f defined in D.

3. If $\mathbf{F} = \nabla f$ then f is called a *potential* of $\mathbf{F}$.

B. Integration Around Closed Curves.

1. $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent in a domain D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ whenever C is a closed path in D. Why is such a field called conservative?

2. Consider the work done by a force F acting along a path on the real line from 0 to d. Using the fact that $F = ma$ where m is mass and a is acceleration, we can deduce that

$$Work = Fd = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2$$

where v_0 is velocity of the particle being acted on by the force when it is at 0, and v_1 is its velocity when it is at position d. That is, work is the net change in potential energy of a particle.

3. In terms of vector forces, the work done by $\mathbf{F}$ along a path given by $\mathbf{r}(t)$, $a \leq t \leq b$, is

$$Work = \int_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2}m|\mathbf{v}(b)|^2 - \frac{1}{2}m|\mathbf{v}(a)|^2$$

where we have assumed that $\mathbf{F} = m\mathbf{a}$ where $\mathbf{a}$ is acceleration, and where $\mathbf{v}(t) = \mathbf{r}'(t)$ is velocity.

4. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent then there is no change in kinetic energy in displacing an object along a loop within the force. No work is done, and the force $\mathbf{F}$ conserves energy.

C. Exact Differentials.

1. We can write
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (F_1\,dx + F_2\,dy + F_3\,dz).$$
The differential $F_1\,dx + F_2\,dy + F_3\,dz$ is *exact* if it is the total differential of some scalar field f, that is, if for some f,
$$F_1\,dx + F_2\,dy + F_3\,dz = df.$$

2. Recall that the total differential of a scalar field f is given by
$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz$$
so that $F_1\,dx + F_2\,dy + F_3\,dz$ is an exact differential if and only if $\mathbf{F} = [F_1, F_2, F_3]$ satisfies $\mathbf{F} = \nabla f$.

3. If $F_1\,dx + F_2\,dy + F_3\,dz$ is exact then $\operatorname{curl}\mathbf{F} = \operatorname{curl}(\nabla f) = \mathbf{0}$.

4. If $\operatorname{curl}\mathbf{F} = \mathbf{0}$ in a domain D and if D has no holes (that is, D is *simply connected*) then the form $F_1\,dx + F_2\,dy + F_3\,dz$ is exact. This means that
$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$

5. Example 4 in the book is an example of a vector field $\mathbf{F}$ such that $\operatorname{curl}\mathbf{F} = \mathbf{0}$ on a domain D but for which $\int_C \mathbf{F} \cdot d\mathbf{r}$ is not zero along every closed path. This is because the domain $D = \{(x, y, z) \colon 0 < x^2 + y^2 < 1,\ z = 0\}$ is not simply connected (it has a hole at the origin).