Quiz 4 Thursday 2.4, 2.6
Exam 1 – Tuesday 9/27 coverage is on line
Continue with Section 3.1

Solving $ay'' + by' + cy = 0$ $y(t_0) = y_0$ $y'(t_0) = y_0'$

Example $y'' - y = 0$ Try: $y = e^{rt}$

$y' = re^{rt}$

$y'' = r^2 e^{rt}$

$r^2 e^{rt} - e^{rt} = 0$

$(r^2 - 1)\cancel{e^{rt}} = 0$ $\Rightarrow$ $y_1(t) = e^{t}$ $(r = 1)$

$r = \pm 1$ $y_2(t) = e^{-t}$ $(r = -1)$.

In general: $ay'' + by' + cy = 0$

$y = e^{rt}$, $y' = re^{rt}$, $y'' = r^2 e^{rt}$

$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$

$(ar^2 + br + c)\cancel{e^{rt}} = 0$

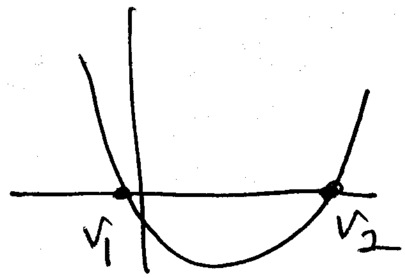
$ar^2 + br + c = 0$. $\leftarrow$ quadratic equation with <u>real</u> coefficients a, b, c.

Possible solutions:

(1) 2 distinct real roots

$(b^2-4ac>0)$ $r=r_1, r_2$

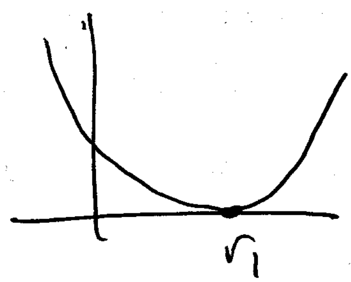
$ar^2+br+c=a(r-r_1)(r-r_2)$



(2) 1 repeated root

$(b^2-4ac=0)$ $r=r_1$

$ar^2+br+c=a(r-r_1)^2$

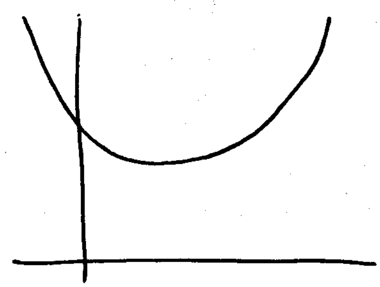


(3) 2 complex roots.

$r=r_1$ $\quad r=\overline{r_1}$

$=x+iy$ $\quad =x-iy$

$(b^2-4ac<0)$



Look at case (1).

e.g. $y''+5y'+6y=0$ $\quad y(0)=2$ $\quad y'(0)=3$.

$y=e^{rt}$ $\quad y'=re^{rt}$ $\quad y''=r^2e^{rt}$

$r^2e^{rt}+5re^{rt}+6e^{rt}=0$

$(r^2+5r+6)\cancel{e^{rt}}=0$

r^2+5r+6 (characteristic polynomial)

Solve $r^2+5r+6=0$

$(r+2)(r+3)=0 \rightarrow r_1=-2 \quad r_2=-3.$

Solutions: $y_1(t) = e^{-2t}$ $\quad y_2(t) = e^{-3t}$

General solution: $y(t) = c_1 e^{-2t} + c_2 e^{-3t}$

$2 = c_1 + c_2 \leftarrow y(0) = 2$

$y'(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}$

$3 = -2c_1 - 3c_2 \leftarrow y'(0) = 3.$

$$\begin{array}{r} 2(c_1 + c_2 = 2) \\ -2c_1 - 3c_2 = 3 \\ \hline -c_2 = 7 \end{array}$$

$c_2 = -7$

$c_1 - 7 = 2 \Rightarrow c_1 = 9$

solution:

$y(t) = 9e^{-2t} - 7e^{-3t}$ //

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e.g. $2y'' + y' - 4y = 0 \quad y(0) = 0 \quad y'(0) = 1$

Solve $2r^2 + r - 4 = 0$

$r = \dfrac{-1 \pm \sqrt{1+32}}{4} = \dfrac{-1 \pm \sqrt{33}}{4}$

$r_1 = \dfrac{-1+\sqrt{33}}{4}$

$r_2 = \dfrac{-1-\sqrt{33}}{4}$

General solution:

$y(t) = c_1 e^{\left(\frac{-1+\sqrt{33}}{4}\right)t} + c_2 e^{\left(\frac{-1-\sqrt{33}}{4}\right)t}$

$y(0) = 0 \qquad 0 = c_1 + c_2$ ✓

$y'(t) = \left(\frac{-1+\sqrt{33}}{4}\right) c_1 e^{(\ )t} + \left(\frac{-1-\sqrt{33}}{4}\right) c_2 e^{(\ )t}$

$1 = \dfrac{-1+\sqrt{33}}{4} c_1 + \dfrac{-1-\sqrt{33}}{4} c_2$ ✓

$$0 = c_1 + c_2 \rightarrow c_2 = -c_1$$

$$1 = \frac{-1+\sqrt{33}}{4} c_1 - \left(\frac{-1-\sqrt{33}}{4}\right) c_1$$

$$= c_1 \left(\frac{\cancel{-1} + \sqrt{33} - (\cancel{-1} - \sqrt{33})}{4} \right) = c_1 \cdot \frac{\sqrt{33}}{2}.$$

$$c_1 = \frac{2}{\sqrt{33}} \quad c_2 = \frac{-2}{\sqrt{33}}$$

$$y(t) = \frac{2}{\sqrt{33}} e^{\left(\frac{-1+\sqrt{33}}{4}\right)t} - \frac{2}{\sqrt{33}} e^{\left(\frac{-1-\sqrt{33}}{4}\right)t}$$

3.2 The Wronskian.

Goal: To understand the structure of solutions to $ay'' + by' + cy = 0$, rather than specific solution technique.

Questions: In previous example we had to solve $\begin{array}{l} c_1 + c_2 = 2 \\ -2c_1 - 3c_2 = 3 \end{array}$. How did we know there would be a solution?

$$\begin{pmatrix} 1 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \longrightarrow$$

Has a unique solution if and only if $\det\begin{pmatrix} 1 & 1 \\ -2 & -3 \end{pmatrix} \neq 0$,

$$\det\begin{pmatrix} 1 & 1 \\ -2 & -3 \end{pmatrix} = (1)(-3) - (1)(-2) = -1 \neq 0.$$

Does a solution always exist?

If it exists, is it unique?

For which t do solutions exist?

Differential operator notation

$L(\varphi) = \varphi''(t) + p(t)\varphi'(t) + q(t)\varphi(t)$ ← differential operator.

When we solve

$$y'' + p(t)y' + q(t)y = 0$$

we are solving: $L(\varphi) = 0$, or $L(y) = 0$.

Thm 3.2.1: Solution to $L[y] = g(t)$ (i) exists

$y(t_0) = y_0$, $y'(t_0) = y'_0$

(ii) is unique

and exists for all t on which $p(t), q(t)$ and $g(t)$ are continuous.

e.g. #9 $t(t-4)y'' + 3ty' + 4y = 2$

$$y(3) = 0 \quad y'(3) = -1$$

Find all t for which the solution exists.

$$L[y] = y'' + \underbrace{\frac{3\cancel{t}}{\cancel{t}(t-4)}}_{p(t)} y' + \underbrace{\frac{4}{t(t-4)}}_{q(t)} y = \underbrace{\frac{2}{t(t-4)}}_{g(t).}$$

$p(t)$ cont on $(-\infty, 4) \cup (4, \infty)$

$q(t)$ cont on $(-\infty, 0) \cup (0, 4) \cup (4, \infty)$

$g(t)$ cont on $(-\infty, 0) \cup (0, 4) \cup (4, \infty)$.

Since $3 \in (0, 4)$ solution exists for $t \in (0, 4)$.

What about the first question?

1. Principle of Superposition

   If we are solving $L[y]=0$ and if y_1 and y_2 satisfy $L[y_1]=0$, $L[y_2]=0$. Then so does $L[c_1y_1+c_2y_2]=0$ for c_1, c_2 constants.

2. When we solve $L[y]=0$ we look for 2 solutions $y_1(t)$, $y_2(t)$ then we try to solve

$$c_1 y_1(t_0) + c_2 y_2(t_0) = y_0$$
$$c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0'$$

for c_1, c_2.

Here $y(t_0)=y_0$, $y'(t_0)=y_0'$.

3. This ~~eq~~ system will have unique solution if and only if

$$\det\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0).$$

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

$\| \; W(y_1,y_2)(t_0)$

Wronskian of y_1 and y_2.

e.g. if $y_1(x) = x$ $\quad y_2(x) = xe^x$

find $W(y_1, y_2)(x)$.

$$W(y_1,y_2)(x) = \det \begin{array}{cc} y_1 & y_2 \\ \left(\begin{array}{c} x \\ 1 \end{array}\right. & \left.\begin{array}{c} xe^x \\ xe^x + e^x \end{array}\right) \\ y_1' & y_2' \end{array}$$

$$= x(xe^x + e^x) - (xe^x)(1) = x^2e^x + xe^x - xe^x$$

$$= x^2 e^x.$$

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1'.$$

e.g. $y'' - y = 0$ $\quad y_1 = e^t$ $\quad y_2 = e^{-t}$

$$W(y_1, y_2)(t) = y_1 y_2' - y_2 y_1'$$

$$= e^t(-e^{-t}) - (e^{-t})(e^t)$$

$$= -1 - 1 = -2 \longleftarrow \neq 0 \text{ for all } t.$$

Abel's Theorem

If y_1, y_2 are solutions to $L[y] = 0$

$\hookrightarrow y'' + \underline{p(t)}y' + q(t)y$

then $W(y_1, y_2)$ satisfies

$$W(y_1, y_2) = c\, e^{\left(-\int p(t)\,dt\right)}$$

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eg $y'' - y = 0$ $\quad p(t) = 0$ $\quad \boxed{q(t) = 1}$

$$W(y_1, y_2) = c\, e^{-\int 0\,dt} = c\, e^{k = \text{const}}$$

$$= \text{const.}$$