

Answer all of the following questions in the space provided. Show all work as partial credit may be given. Answers without justification, even if they are correct, will earn no credit.

1. (2 pts. each) Let $f(x, y) = y/x^2$.

(a) Find ∇f , the gradient of f .

$$\begin{aligned}\nabla f &= f_x \vec{i} + f_y \vec{j} \\ &= -\frac{2y}{x^3} \vec{i} + \frac{1}{x^2} \vec{j} \quad //\end{aligned}$$

(b) Find the directional derivative of $f(x, y)$ at the point $(2, 3)$ in the direction $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$.

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{i} + 2\vec{j}}{\sqrt{5}} = \frac{1}{\sqrt{5}} \vec{i} + \frac{2}{\sqrt{5}} \vec{j}$$

$$\begin{aligned}D_{\vec{u}} f(2, 3) &= \nabla f(2, 3) \cdot \vec{u} = \left(-\frac{3}{4} \vec{i} + \frac{1}{4} \vec{j}\right) \cdot \left(\frac{1}{\sqrt{5}} \vec{i} + \frac{2}{\sqrt{5}} \vec{j}\right) \\ &= -\frac{3}{4\sqrt{5}} + \frac{2}{4\sqrt{5}} = -\frac{1}{4\sqrt{5}} \quad //\end{aligned}$$

(c) Find the maximum rate of change of $f(x, y)$ at the point $(2, 3)$.

$$\begin{aligned}\text{Max rate of change is } & |\nabla f(2, 3)| \\ &= \left| -\frac{3}{4} \vec{i} + \frac{1}{4} \vec{j} \right| = \frac{1}{4} (3^2 + 1^2)^{1/2} = \frac{\sqrt{10}}{4} \quad //\end{aligned}$$

2. (4 pts.) If $z = \ln(x - y)$, $x = r \cos(\theta)$, $y = r \sin(\theta)$, find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$. Be sure to write each derivative as a function of r and θ .

$$\begin{aligned}\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r} = \frac{1}{x-y} \cos \theta - \frac{1}{x-y} \sin \theta \\ &= \frac{\cos \theta - \sin \theta}{r \cos \theta - r \sin \theta} = \frac{1}{r} \quad //\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial \theta} = \frac{1}{x-y} \cdot -r \sin \theta - \frac{1}{x-y} \cdot r \cos \theta \\ &= \frac{-r(\sin \theta + \cos \theta)}{r \cos \theta - r \sin \theta} = \frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \quad //\end{aligned}$$