MATH 213 - EXAM 2 - SOLUTIONS

1. $\vec{r}(t) = 2t\,\vec{i} + t^2\,\vec{j}$

(a) $\vec{r}\,'(t) = 2\vec{i} + 2t\vec{j}$

$|\vec{r}\,'(t)| = (2^2 + (2t)^2)^{1/2} = 2(1+t^2)^{1/2}$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{1}{(1+t^2)^{1/2}}\vec{i} + \frac{t}{(1+t^2)^{1/2}}\vec{j} \quad //$$

(b) $\vec{T}(t) = (1+t^2)^{-1/2}(\vec{i} + t\vec{j})$

$$\therefore \vec{T}\,'(t) = (1+t^2)^{-1/2}(\vec{j}) + (-\tfrac{1}{\cancel{2}})(1+t^2)^{-3/2}(\cancel{2}t)(\vec{i} + t\vec{j})$$

$$= \frac{-t}{(1+t^2)^{3/2}}\vec{i} + \left[\frac{1}{(1+t^2)^{1/2}} - \frac{t^2}{(1+t^2)^{3/2}}\right]\vec{j}$$

$$= \frac{-t}{(1+t^2)^{3/2}}\vec{i} + \left[\frac{1+t^2}{(1+t^2)^{3/2}} - \frac{t^2}{(1+t^2)^{3/2}}\right]\vec{j}$$

$$= \frac{1}{(1+t^2)^{3/2}}(-t\vec{i} + \vec{j})$$

$$\therefore |\vec{T}\,'(t)| = \frac{1}{(1+t^2)^{3/2}}(t^2+1)^{1/2}$$

$$\therefore K(t) = |\vec{T}\,'(t)| / |\vec{r}\,'(t)| = \frac{1}{(1+t^2)^{3/2}}\cancel{(1+t^2)^{1/2}} \cdot \frac{1}{2\cancel{(1+t^2)^{1/2}}}$$

$$= \frac{1}{2(1+t^2)^{3/2}} \quad //$$

2. (a) $f(x,y) = x^2 - xy^2 + 2y^3$

$\frac{\partial f}{\partial x} = 2x - y^2 \quad \frac{\partial f}{\partial y} = -2xy + 6y^2$ //

(b) $f(x,y) = \frac{x^2}{x^3+y^3}$

$\frac{\partial f}{\partial x} = \frac{2x(x^3+y^3) - x^2(3x^2)}{(x^3+y^3)^2} = \frac{2xy^3 - x^4}{(x^3+y^3)^2}$ //

$\frac{\partial f}{\partial y} = \frac{0 - x^2(3y^2)}{(x^3+y^3)^2} = \frac{-3x^2y^2}{(x^3+y^3)^2}$ //

(c) $f(x,y) = e^{-x^2}\cos(x^2-y)$

$\frac{\partial f}{\partial x} = e^{-x^2}(-\sin(x^2-y)\cdot 2x) + (-2x)e^{-x^2}\cos(x^2-y)$

$= -2xe^{-x^2}\sin(x^2-y) - 2xe^{-x^2}\cos(x^2-y)$ //

$\frac{\partial f}{\partial y} = -e^{-x^2}\sin(x^2-y)(-1) = e^{-x^2}\sin(x^2-y)$ //

3. (a) $f(x,y) = x^4 + 2x^2y + y^2$

$f_x = 4x^3 + 4xy \quad f_{xx} = 12x^2 + 4y$

$f_y = 2x^2 + 2y \quad f_{yy} = 2$

$f_{xy} = 4x = f_{yx}$

(b) $f(x,y) = \sin(3xy)$

$f_x = 3y\cos(3xy)$ $\quad f_y = 3x\cos(3xy)$

$f_{xx} = -9y^2\sin(3xy)$ $\quad f_{yy} = -9x^2\sin(3xy)$ //

$f_{xy} = 3y(-3x\sin(3xy)) + 3\cos(3xy)$

$= -9xy\sin(3xy) + 3\cos(3xy)$ //

$= f_{yx}$

4. $f(x,y,z) = e^{xyz}$

$f_z = xy\, e^{xyz}$

$f_{zx} = (xy)(yz\, e^{xyz}) + y\, e^{xyz}$

$= (xy^2z + y)e^{xyz}$

$f_{zxz} = (xy^2z + y)(xy)e^{xyz} + xy^2 e^{xyz}$

$= (x^2y^3z + 2xy^2)e^{xyz}$ //.

5. $V = \pi r^2 h$

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$

$$= 2\pi r h \frac{dr}{dt} + \pi r^2 \frac{dh}{dt} \quad //$$

6. $f(x,y,z) = xy + yz^2 + xz^3$

(a) $\nabla f = (y+z^3)\vec{i} + (x+z^2)\vec{j} + (2yz+3xz^2)\vec{k}$

(b) $\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 2,-1,2\rangle}{3} = \langle \frac{2}{3}, \frac{-1}{3}, \frac{2}{3} \rangle$

$\nabla f(1,0,2) = 8\vec{i} + 5\vec{j} + 12\vec{k}$

$$D_{\vec{u}} f(1,0,2) = 8\left(\tfrac{2}{3}\right) + 5\left(-\tfrac{1}{3}\right) + 12\left(\tfrac{2}{3}\right)$$

$$= \frac{35}{3} \quad //.$$

(c) max rate of change is

$$|\nabla f(1,0,2)| = (64+25+144)^{1/2} = \sqrt{233} \quad //$$