MATH 213 - EXAM 1 - SOLUTIONS

1. (a) $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

$\vec{a}\cdot\vec{b} = (6\vec{i} - 5\vec{j} + \vec{k})\cdot(\vec{i} + \vec{k}) = 6 + 1 = 7$

$|\vec{a}| = (36 + 25 + 1)^{1/2} = \sqrt{62}$

$|\vec{b}| = (1+1)^{1/2} = \sqrt{2}$

$\therefore \cos\theta = \dfrac{7}{\sqrt{62}\cdot\sqrt{2}} \approx .6286$

$\therefore \theta \approx 51^\circ$

(b) unit vector $= \dfrac{\vec{b}\times\vec{c}}{|\vec{b}\times\vec{c}|} = -\dfrac{1}{3\sqrt{2}}\vec{i} + \dfrac{4}{3\sqrt{2}}\vec{j} + \dfrac{1}{3\sqrt{2}}\vec{k}$ //

$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 1 & 1 & -3 \end{vmatrix} = -\vec{i} + 4\vec{j} + \vec{k}$

$|\vec{b}\times\vec{c}| = (1 + 16 + 1)^{1/2} = \sqrt{18} = 3\sqrt{2}$

(c) $\text{proj}_{\vec{b}}(\vec{a}) = \left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b} = \dfrac{7}{2}\vec{b}$

$= \dfrac{7}{2}\vec{i} + \dfrac{7}{2}\vec{k}$ //

(d) $\vec{c} = \vec{i} + \vec{j} - 3\vec{k}$

$$x = 6 + t$$
$$y = -5 + t$$
$$z = 1 - 3t \quad //$$

(e) $\vec{a} = 6\vec{i} - 5\vec{j} + \vec{k}$

$$6(x-1) - 5(y-1) + (z+3) = 0$$
$$6x - 5y + z - 6 + 5 + 3 = 0$$
$$6x - 5y + z = -2 \quad //$$

(f) $\vec{a} \cdot (\vec{b} \times \vec{c}) = (6\vec{i} - 5\vec{j} + \vec{k}) \cdot (-\vec{i} + 4\vec{j} + \vec{k})$

$$= -6 - 20 + 1 = -25$$

$\therefore \ |\vec{a} \cdot (\vec{b} \times \vec{c})| = 25 \ //$

2. (a) $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -4 & -4 \\ 1 & -5 & -3 \end{vmatrix}$

$= -8\vec{i} - 10\vec{j} + 14\vec{k}$ plane contains $(0,0,1)$

$$-8x - 10y + 14(z-1) = 0$$
$$-8x - 10y + 14z = 14 \ //$$

(b) $\text{Area} = \frac{1}{2}\left|\vec{AB} \times \vec{AC}\right|$

$= \frac{1}{2}\left|-8\vec{i} - 10\vec{j} + 14\vec{k}\right|$

$= \frac{1}{2}(64 + 100 + 196)^{1/2}$

$= \frac{1}{2}\sqrt{360} = 3\sqrt{10}$ //

3. $\vec{v} = (\vec{i} - \vec{j} + \vec{k}) \times (2\vec{i} + \vec{j} + \vec{k})$

$= (\vec{i} \times \vec{j}) + (\vec{i} \times \vec{k}) - 2(\vec{j} \times \vec{i}) - (\vec{j} \times \vec{k}) + 2(\vec{k} \times \vec{i}) + (\vec{k} \times \vec{j})$

$= \vec{k} - \vec{j} + 2\vec{k} - \vec{i} + 2\vec{j} - \vec{i}$

$= -2\vec{i} + \vec{j} + 3\vec{k}$ //

point: Let $x = 0$

$-y + z = 1$

$y + z = 2$

$2z = 3$

$z = \frac{3}{2}$

$y = \frac{1}{2}$

$(0, \frac{1}{2}, \frac{3}{2})$ //

$\therefore \quad x = -2t$

$y = \frac{1}{2} + t$

$z = \frac{3}{2} + 3t$ //

4. (a) $\vec{r}(t) = (t-\sin t)\vec{i} + (1-\cos t)\vec{j} + t\vec{k}$

$\vec{v}(t) = \vec{r}\,'(t) = (1-\cos t)\vec{i} + \sin t\,\vec{j} + \vec{k}$

$\vec{a}(t) = \vec{r}\,''(t) = \sin t\,\vec{i} + \cos t\,\vec{j}$

speed $= |\vec{v}(t)| = |\vec{r}\,'(t)|$

$= ((1-\cos t)^2 + \sin^2 t + 1)^{1/2}$

$= (1 - 2\cos t + \cos^2 t + \sin^2 t + 1)^{1/2}$

$= (3 - 2\cos t)^{1/2}$

(b) point: $\vec{r}(0) = 0\vec{i} + 0\vec{j} + 0\vec{k}$

$(0,0,0)$

$\vec{v} = \vec{r}\,'(0) = 0\vec{i} + 0\vec{j} + \vec{k}$

$x = 0$

$y = 0$

$z = t$ //

5. $\vec{r}(t) = (2t+3)\vec{i} + (5-t^2)\vec{j} + 3t\vec{k}, \quad 1 \le t \le 4$

(a) $\vec{r}\,'(t) = 2\vec{i} - 2t\vec{j} + 3\vec{k}$

$|\vec{r}\,'(t)| = (4 + 4t^2 + 9)^{1/2} = (13 + 4t^2)^{1/2}$

$\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{2}{(13+4t^2)}\vec{i} - \dfrac{2t}{(13+4t^2)^{1/2}}\vec{j} + \dfrac{3}{(13+4t^2)^{1/2}}\vec{k}$ //

(b) $L = \int_1^4 |\vec{r}\,'(t)|\,dt = \int_1^4 (13+4t^2)^{1/2}\,dt$ //