12-9-09

Review Session – Friday 9–$10^{\underline{15}}$

Final Exam – Monday 7^{30}_{a} – 10^{15}_{a}

Fill out course evaluations online

1. Differential form of a line integral.

C: curve in space, $\vec{r}(t)$, $a \le t \le b$ f – function of ~~two~~ 3 variables

$$\int_C f\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

C: curve in space, $\vec{r}(t)$, $a \le t \le b$.

$\vec{F}$: vector field $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$

$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

$$\int_C (\vec{F}\cdot\vec{T})\,ds = \int_a^b \left(\vec{F}(\vec{r}(t))\cdot \frac{\vec{r}\,'(t)}{\cancel{|\vec{r}\,'(t)|}}\right) \cancel{|\vec{r}\,'(t)|}\,dt$$

$\vec{T}$: unit tangent vector to C

$$= \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

$$= \int_a^b \left(\vec{F}\cdot\frac{d\vec{r}}{dt}\right) dt$$

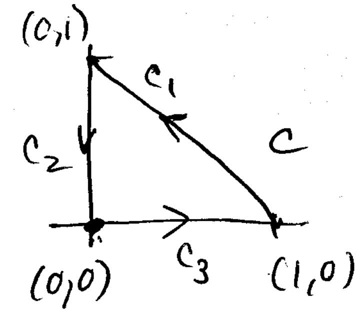
$$= \int_C \vec{F}\cdot d\vec{r}.$$

$$= \int_C M\,dx + N\,dy + P\,dz$$

differential form of line integral

$$\boxed{\begin{aligned} \underline{\vec{F}\cdot d\vec{r}} &= \left(\vec{F}\cdot\frac{d\vec{r}}{dt}\right) dt \\ &= (M\vec{i} + N\vec{j} + P\vec{k})\cdot\left(\frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}\right) dt \\ &= \left(M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt}\right) dt \\ &= M\,dx + N\,dy + P\,dz \end{aligned}}$$

16.2 #18) $\int_C (x-y)\,dx + (x+y)\,dy$



$C_1 : (1-t)\langle 1,0\rangle + t\langle 0,1\rangle \quad 0 \le t \le 1$

$= \langle 1-t, 0\rangle + \langle 0, t\rangle$

$= \langle 1-t, t\rangle$

$\boxed{\begin{array}{c} x = 1-t \\ y = t \\ 0 \le t \le 1 \end{array}}$

$C_2 : (1-t)\langle 0,1\rangle + t\langle 0,0\rangle$

$= \langle 0, 1-t\rangle$

$\boxed{\begin{array}{c} x = 0 \\ y = 1-t \\ 0 \le t \le 1 \end{array}}$

$C_3 : (1-t)\langle 0,0\rangle + t\langle 1,0\rangle$

$= \langle t, 0\rangle$

$\boxed{\begin{array}{c} x = t \\ y = 0 \\ 0 \le t \le 1 \end{array}}$

$\int_{C_1} (x-y)\,dx + (x+y)\,dy$

$= \int_0^1 \left((x-y)\frac{dx}{dt} + (x+y)\frac{dy}{dt}\right) dt$

$= \int_0^1 \left((1-t-t)(-1) + (1-t+t)(1)\right) dt$

$= \int_0^1 2t-1+1\,dt = \int_0^1 2t\,dt = t^2\Big|_0^1$

$= 1$

$\int_{C_2} (x-y)\,dx + (x+y)\,dy$

$= \int_0^1 \left((0-(1-t))(0) + (0+1-t)(-1)\right) dt$

$= \int_0^1 t-1\,dt = \frac{1}{2}t^2 - t\Big|_0^1 = \frac{-1}{2}$

$\int_{C_3} (x-y)\,dx + (x+y)\,dy$

$= \int_0^1 \left(t(1) + t(0)\right) dt$

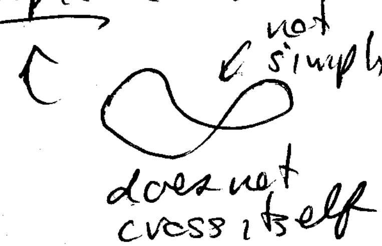
$= \int_0^1 t\,dt = \frac{1}{2}t^2\Big|_0^1 = \frac{1}{2}$

$\int_C (x-y)\,dx + (x+y)\,dy$

$= 1 - \frac{1}{2} + \frac{1}{2} = 1$

16.4 Greens Thm in the Plane

Idea: Consider a region R in the plane surrounded (or bounded) by a simple closed curve C.



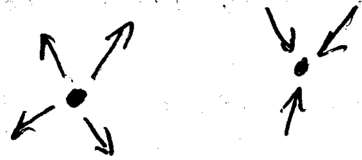
Think of a vector field $\vec{F}$ as a velocity field for some fluid.

We measure two things:

(a) tendency of the fluid to flow away ~~at~~ or toward a point.

(b) tendency of the fluid to flow around a point (i.e. to circulate)

(a)



Divergence of $\vec{F}$,

$$\text{div}\,\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

$(\vec{F} = M\hat{i} + N\hat{j} + P\hat{k})$

NB: div $\vec{F}$ is a scalar quantity, ~~i.e.~~ that is, a function of (x, y, z).

(b) Curl of $\vec{F}$

$$\text{curl } \vec{F} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k} \quad \text{if } \vec{F} = M\vec{i} + N\vec{j}$$

NOTE: curl $\vec{F}$ is a vector quantity

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} \qquad \nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

$$\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$$

$$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left(M\vec{i} + N\vec{j} + P\vec{k}\right)$$

$$= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = \text{div } \vec{F}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\vec{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\vec{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}$$

$$+\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)$$

$$= \text{curl } \vec{F}$$

Green's Thm

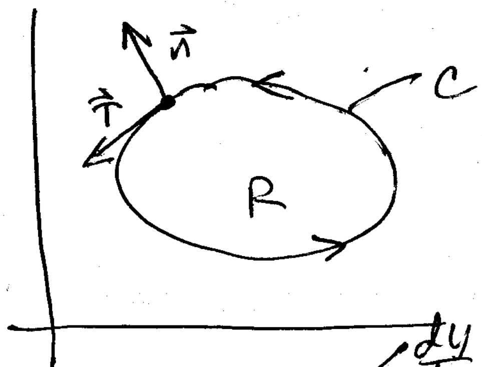
1. $$\int_C (\vec{F} \cdot \vec{n})\, ds = \iint_R (\text{div } \vec{F})\, dx\, dy$$

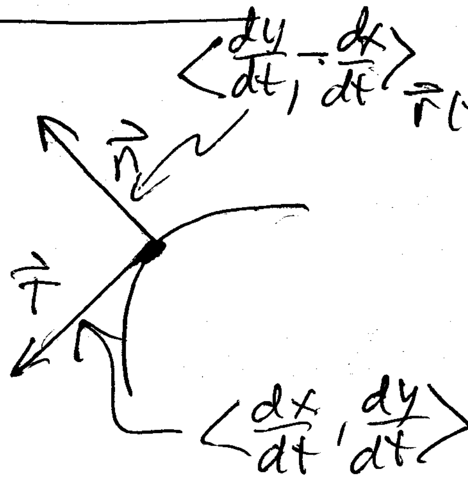
2. $$\int_C (\vec{F} \cdot \vec{T})\, ds = \iint_R (\text{curl } \vec{F} \cdot \vec{k})\, dx\, dy$$

$$\oint_C (\vec{F}\cdot\vec{n})\,ds = \text{flux integral}$$

unit outward normal

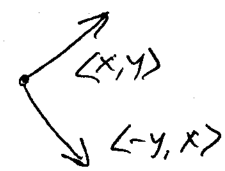
$$= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$$





$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$$

$$\vec{T} = \frac{\left(\frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}\right)}{|\vec{r}\,'(t)|}$$

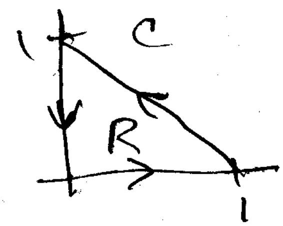


$$\int_C \vec{F}\cdot\vec{n}\,ds = \int_C M\,dy - N\,dx$$

$$\boxed{\int_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy.}$$

$$\int_C (\vec{F}\cdot\vec{T})\,ds = \boxed{\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.}$$

e.g. $\int_C (x-y)\,dx + (x+y)\,dy$



$M = x-y \quad N = x+y$

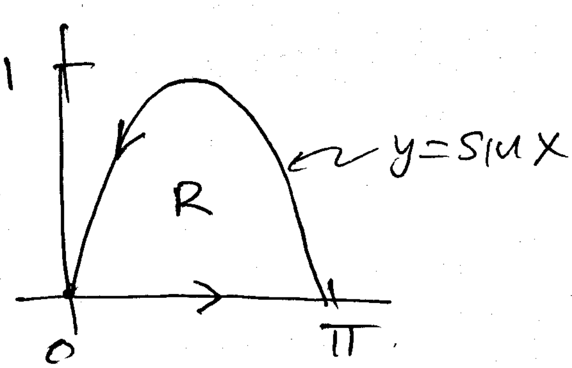
$\frac{\partial M}{\partial y} = -1 \quad \frac{\partial N}{\partial x} = 1$

$$= \iint_R (1-(-1))\,dx\,dy$$

$$= \iint_R 2\,dx\,dy = 2\cdot(\text{area of } \Delta)$$

$$= 2\cdot\frac{1}{2} = 1$$

#(8) $\int_C 3y\,dx + 2x\,dy$



use flux form of Green:

$M = 2x \quad N = -3y$

$\frac{\partial M}{\partial x} = 2 \quad \frac{\partial N}{\partial y} = -3$

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) \underbrace{dx\,dy}_{dA} = \iint_R (2-3)\,dx\,dy$$

$$= \iint_R (-1)\,dx\,dy = \int_0^{\pi} \int_0^{\sin x} (-1)\,dy\,dx \quad \text{etc}\cdots$$