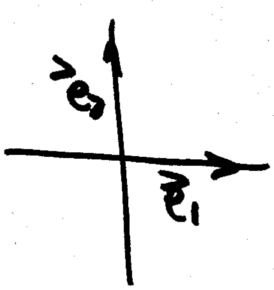
5.4 Eigenvectors + Linear Transformations

1. $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation

then $T(\vec{x}) = A\vec{x}$ A is $m \times n$

$$A = \left[T(\vec{e}_1)\ T(\vec{e}_2)\ \cdots\ T(\vec{e}_n) \right]$$



$T: V \to W$ $\dim V = n$

T is linear $\dim W = m$

Point: T can be represented by a matrix M using coordinate vectors.

How? (a) Fix $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ basis for V

+ $\mathcal{C} = \{\vec{c}_1, \ldots, \vec{c}_m\}$ basis for W

(b) Diagram:

$$
\begin{tikzcd}
\vec{x} & V \arrow[r, "T"] \arrow[d, "{[\cdot]_B}"'] & W \arrow[d, "{[\cdot]_C}"] & T\vec{x} \\
{[\vec{x}]_B} & \mathbb{R}^n \arrow[r] & \mathbb{R}^m & {[T\vec{x}]_C}
\end{tikzcd}
$$

$M \leftarrow$ what matrix?

$\vec{x} \in V \quad \vec{x} = r_1\vec{b}_1 + \cdots + r_n\vec{b}_n$

$$[\vec{x}]_B = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \qquad \begin{aligned} T\vec{x} &= T(r_1\vec{b}_1 + \cdots + r_n\vec{b}_n) \\ &= r_1T(\vec{b}_1) + \cdots + r_nT(\vec{b}_n) \end{aligned}$$

$$
\begin{aligned}
[T\vec{x}]_C &= [r_1T(\vec{b}_1) + \cdots + r_nT(\vec{b}_n)]_C \\
&= r_1[T(\vec{b}_1)]_C + \cdots + r_n[T(\vec{b}_n)]_C \\
&= \underbrace{\begin{bmatrix} [T(\vec{b}_1)]_C & \cdots & [T(\vec{b}_n)]_C \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}}_{[\vec{x}]_B}
\end{aligned}
$$

Say: M is the matrix representation of T with respect to $\mathcal{B}$ and $\mathcal{C}$.

<u>Example</u>: 1, p329 $\quad T: V \Rightarrow W$

$\{\vec{b}_1, \vec{b}_2\} \quad \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$

$$M = \left[[T(\vec{b}_1)]_{\mathcal{C}} \, [T(\vec{b}_2)]_{\mathcal{C}} \right]$$

$$[T(\vec{b}_1)]_{\mathcal{C}} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \quad [T(\vec{b}_2)]_{\mathcal{C}} = \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}$$

3x2

$$M = \begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix}$$

dimV $\quad$ dimW.

$$\mathbb{R}^{2} \longrightarrow \mathbb{R}^{3}$$

$$\vec{x} \mapsto M\vec{x}$$

e.g. #4 $T: V \to \mathbb{R}^2$ $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$

$$T(x_1\vec{b}_1 + x_2\vec{b}_2 + x_3\vec{b}_3) = \begin{bmatrix} 2x_1 - 4x_2 + 5x_3 \\ -x_2 + 3x_3 \end{bmatrix}$$

$$M = \left[[T(\vec{b}_1)]_{\mathcal{E}} \; [T(\vec{b}_2)]_{\mathcal{E}} \; [T(\vec{b}_3)]_{\mathcal{E}} \right]$$

$$[T(\vec{b}_1)]_{\mathcal{E}} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad [T(\vec{b}_2)]_{\mathcal{E}} = \begin{bmatrix} -4 \\ -1 \end{bmatrix} \quad [T(\vec{b}_3)]_{\mathcal{E}} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$M = \begin{bmatrix} 2 & -4 & 5 \\ 0 & -1 & 3 \end{bmatrix}$$

2. Special case $T: V \to V$

Fix <u>one</u> basis $\mathcal{B}$ for V

We say M is the matrix representation of T with respect to $\mathcal{B}$ say

We write $M = [T]_{\mathcal{B}}$ $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$

$$[T]_{\mathcal{B}} = \left[\, [T(\vec{b}_1)]_{\mathcal{B}} \; [T(\vec{b}_2)]_{\mathcal{B}} \cdots [T(\vec{b}_n)]_{\mathcal{B}} \,\right]$$

e.g #8 $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ basis for V.

$$T(3\vec{b}_1 - 4\vec{b}_2)$$

$$[T]_{\mathcal{B}} = \begin{bmatrix} 0 & -6 & 1 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix}$$

$$\begin{tikzcd}
\vec{x} & V \arrow[r, "T"] \arrow[d, "{[\cdot]_{\mathcal{B}}}"'] & V \arrow[d, "{[\cdot]_{\mathcal{B}}}"] & T\vec{x} \\
{[\vec{x}]_{\mathcal{B}}} & \mathbb{R}^n \arrow[r, "{[T]_{\mathcal{B}}}"'] & \mathbb{R}^n & {[T\vec{x}]_{\mathcal{B}} = [T]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}}
\end{tikzcd}$$

$$\boxed{[T\vec{x}]_{\mathcal{B}} = [T]_{\mathcal{B}}\,[\vec{x}]_{\mathcal{B}}}$$

$T(\vec{x})$ where $\vec{x} = 3\vec{b}_1 - 4\vec{b}_2$

$$[\vec{x}]_B = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$$

$$[T(\vec{x})]_B = [T]_B [\vec{x}]_B = \begin{bmatrix} 0 & -6 & 1 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$$

$3+8$

$$= \begin{bmatrix} 24 \\ -20 \\ 11 \end{bmatrix}$$

$$T(\vec{x}) = 24\vec{b}_1 - 20\vec{b}_2 + 11\vec{b}_3$$

Eigenvalues + Linear Transformations cont'd.

$$\begin{tikzcd} \vec{x} \quad V \arrow[r, "T"] \arrow[d, "{[\cdot]_B}"'] & W \quad T\vec{x} \arrow[d, "{[\cdot]_C}"] \\ \mathbb{R}^n \arrow[r, "M"'] & \mathbb{R}^m \end{tikzcd}$$

$\dim V = n$
$\dim W = m$
$\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ basis for V
$\mathcal{C} = \{\vec{c}_1, \dots, \vec{c}_m\}$ basis for W

M is $m \times n$ matrix

$$M[\vec{x}]_{\mathcal{B}} = [T(\vec{x})]_{\mathcal{C}}$$

$$M = \big[[T(\vec{b}_1)]_{\mathcal{C}} \; [T(\vec{b}_2)]_{\mathcal{C}} \cdots [T(\vec{b}_n)]_{\mathcal{C}} \big]$$

Special case:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "{[\cdot]_B}"'] & V \arrow[d, "{[\cdot]_B}"] \\ \mathbb{R}^n \arrow[r, "{[T]_B}"'] & \mathbb{R}^n \end{tikzcd}$$

$\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ basis for V

$$[T]_{\mathcal{B}} [\vec{x}]_{\mathcal{B}} = [T\vec{x}]_{\mathcal{B}}$$

e.g #6 $T: \mathbb{P}_2 \to \mathbb{P}_4$

$$p(t) \xmapsto[T]{} p(t) + t^2 p(t) = (1+t^2)\,p(t)$$

a. Find $T(2-t+t^2) = (1+t^2)(2-t+t^2)$

$$= 2 - t + 3t^2 - t^3 + t^4$$

b. $T(c\,p(t) + d\,q(t)) = (1+t^2)(c\,p(t) + d\,q(t))$

$$= c\left[(1+t^2)p(t)\right] + d\left[(1+t^2)q(t)\right]$$

$$= c\,T(p(t)) + d\,T(q(t))$$

c. $\mathcal{B} = \{1, t, t^2\}$ for $\mathbb{P}_2$

$\mathcal{C} = \{1, t, t^2, t^3, t^4\}$ for $\mathbb{P}_4$

$$M = \left[\,[T(\vec{b}_1)]_{\mathcal{C}} \cdots [T(\vec{b}_n)]_{\mathcal{C}}\,\right]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$T(1) = (1+t^2)1 = 1+t^2$

$T(t) = t + t^3$

$T(t^2) = t^2 + t^4$

3. Similarity

$T(\vec{x}) = A\vec{x}$

A is $n \times n$.

$$\mathbb{R}^n \xrightarrow[A]{T} \mathbb{R}^n$$

Could say A is the matrix of T relative to $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$, i.e.

$$A = [T]_{\mathcal{E}}$$

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "T"] \arrow[d, "{[\cdot]_{\mathcal{E}}}"'] & \mathbb{R}^n \arrow[d, "{[\cdot]_{\mathcal{E}}}"] \\
\mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^n
\end{tikzcd}$$

Suppose $A = PBP^{-1}$ (A similar to B)

1. $P = [\vec{v}_1 \cdots \vec{v}_n]$

Since P is invertible

$\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$

2.

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "T", "A"'] \arrow[d, "{[\cdot]_B}"', "P^{-1}"] & \mathbb{R}^n \arrow[d, "P^{-1}"', "{[\cdot]_B}"] \\
\mathbb{R}^n \arrow[r, "B"'] & \mathbb{R}^n
\end{tikzcd}$$

$$\vec{x} = \underbrace{[\vec{v}_1 \cdots \vec{v}_n]}_{P} [\vec{x}]_B \quad \therefore \quad P^{-1}\vec{x} = [\vec{x}]_B$$

$$P^{-1}A = B P^{-1}$$

$$\therefore \; A = P B P^{-1}$$

<u>Thm</u>: If A is $n \times n$ matrix and $A = PBP^{-1}$, $P = [\vec{v}_1 \cdots \vec{v}_n]$, and if $T(\vec{x}) = A\vec{x}$ <u>then</u>

$[T]_{\mathcal{B}} = B$ where $\mathcal{B} = \{\vec{v}_1 \cdots \vec{v}_n\}$.

e.g. #14 p334

$$A = \begin{bmatrix} 5 & -3 \\ -7 & 1 \end{bmatrix}$$

Find D so that $A = PDP^{-1}$

D diagonal

Find all eigenvalues / eigenvectors of A.

1. $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 5-\lambda & -3 \\ -7 & 1-\lambda \end{vmatrix} = (5-\lambda)(1-\lambda) - 21$$

$$= \lambda^2 - 6\lambda + 5 - 21 = \lambda^2 - 6\lambda - 16 = 0$$

$$(\lambda + 2)(\lambda - 8) = 0$$

$$\lambda = -2 \quad \lambda = 8.$$

2. $\lambda = -2$ $A - \lambda I = A + 2I = \begin{bmatrix} 7 & -3 \\ -7 & 3 \end{bmatrix}$

$$\text{Nul}(A+2I): \quad \begin{bmatrix} 7 & -3 & 0 \\ -7 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 7 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

x_2

$$\vec{x} = \begin{bmatrix} \frac{3}{7} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3/7 \\ 1 \end{bmatrix}$$

$x_1 = \frac{3}{7} x_2$

x_2 free

Basis for eigenspace: $\left\{ \begin{bmatrix} 3 \\ 7 \end{bmatrix} \right\}$.

$\lambda = 8$ $\quad A - 8I = \begin{bmatrix} -3 & -3 \\ -7 & -7 \end{bmatrix}$

$$\begin{bmatrix} -3 & -3 & 0 \\ -7 & -7 & 0 \end{bmatrix} \sim \begin{bmatrix} -3 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\uparrow$ x_2

$\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ Basis for eigenspace $= \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ $\quad x_1 = -x_2$

$x_2 =$ free

$$A = PDP^{-1}$$

$$P = \begin{bmatrix} 3 & -1 \\ 7 & 1 \end{bmatrix} \quad D = \begin{bmatrix} -2 & 0 \\ 0 & 8 \end{bmatrix}$$

$$\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$