Solutions to Exam 4 will be posted

Recall: Basis : ① linear independence
② span the space

Coordinate vectors $V \longrightarrow \mathbb{R}^n$

4.7 Change of Basis

Let $\beta = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ be a basis for $\mathbb{R}^n$. Then given $\vec{x} \in \mathbb{R}^n$,

$$[\vec{x}]_\beta = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \text{ means } \vec{x} = a_1\vec{b}_1 + a_2\vec{b}_2 + \cdots + a_n\vec{b}_n$$

If we define $P_\beta = [\vec{b}_1 \ \vec{b}_2 \cdots \vec{b}_n]$ then

$$\vec{x} = a_1\vec{b}_1 + \cdots + a_n\vec{b}_n = [\vec{b}_1 \cdots \vec{b}_n]\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = P_\beta [\vec{x}]_\beta$$

So $\vec{x} = P_\beta [\vec{x}]_\beta$ and $[\vec{x}]_\beta = P_\beta^{-1} \vec{x}$

Can also write for $\vec{x}$, $[\vec{x}]_{\mathcal{E}}$ where

$$\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\} \qquad \vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow i \\ \\ \\ \\ \end{matrix}$$

This means

$$[\vec{x}]_{\mathcal{E}} = P_\beta [\vec{x}]_\beta \qquad [\vec{x}]_\beta = P_\beta^{-1} [\vec{x}]_{\mathcal{E}}$$

Conclusions:

1. P_β maps β-coordinate vector of $\vec{x}$ to the $\mathcal{E}$-coordinate vector of $\vec{x}$
2. P_β^{-1} maps $\mathcal{E}$-coordinate vector of $\vec{x}$ to the β-coordinate vector of $\vec{x}$.

Write $P_\beta = \underset{\mathcal{E} \leftarrow \beta}{P} \qquad P_\beta^{-1} = \underset{\beta \leftarrow \mathcal{E}}{P}$

1. Suppose we have 2 bases for V.

$\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ $\quad \mathcal{C} = \{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$

If we have $[\vec{x}]_{\mathcal{B}}$, how can we find $[\vec{x}]_{\mathcal{C}}$?

$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ means $\vec{x} = a_1\vec{b}_1 + a_2\vec{b}_2 + \cdots + a_n\vec{b}_n$

$$\therefore [\vec{x}]_{\mathcal{C}} = [a_1\vec{b}_1 + \cdots + a_n\vec{b}_n]_{\mathcal{C}}$$

$$= a_1[\vec{b}_1]_{\mathcal{C}} + a_2[\vec{b}_2]_{\mathcal{C}} + \cdots + a_n[\vec{b}_n]_{\mathcal{C}}$$

$$= \begin{bmatrix} [\vec{b}_1]_{\mathcal{C}} & \cdots & [\vec{b}_n]_{\mathcal{C}} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$= \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [\vec{x}]_{\mathcal{B}}$$

↑ Key point:
Coordinate mapping
$[\cdot]_{\mathcal{C}} : V \longrightarrow \mathbb{R}^n$
$\vec{x} \longmapsto [\vec{x}]_{\mathcal{C}}$
is linear

Also $[\vec{x}]_{\mathcal{B}} = \underset{C \leftarrow \mathcal{B}}{P}^{-1} [\vec{x}]_{C}$

$\underset{\mathcal{B} \leftarrow C}{P} = \left[[\vec{c}_1]_{\mathcal{B}} \; [\vec{c}_2]_{\mathcal{B}} \cdots [\vec{c}_n]_{\mathcal{B}} \right]$

e.g. Spse $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ $C = \{\vec{c}_1, \vec{c}_2\}$ are bases for V where

$\vec{b}_1 = 4\vec{c}_1 + \vec{c}_2$ $\quad \vec{b}_2 = -6\vec{c}_1 + \vec{c}_2$

If $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ find $[\vec{x}]_{C}$!

Need: $\underset{C \leftarrow \mathcal{B}}{P} = \left[[\vec{b}_1]_C \; [\vec{b}_2]_C \right]$

$[\vec{b}_1]_C = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ $\quad [\vec{b}_2]_C = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$

$\underset{C \leftarrow \mathcal{B}}{P} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix}$

$[\vec{x}]_C = \underset{C \leftarrow \mathcal{B}}{P} [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

This means $6\vec{c}_1 + 4\vec{c}_2 = 3\vec{b}_1 + \vec{b}_2 = \vec{x}$.

We have $\underset{C \leftarrow B}{P} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix}$

What is $\underset{B \leftarrow C}{P}$? $\underset{B \leftarrow C}{P} = \underset{C \leftarrow B}{P}^{-1} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix}^{-1}$

$= \frac{1}{10}\begin{bmatrix} 1 & 6 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & \frac{3}{5} \\ -\frac{1}{10} & \frac{2}{5} \end{bmatrix}$

So $[\vec{c}_1]_B = \begin{bmatrix} \frac{1}{10} \\ -\frac{1}{10} \end{bmatrix}$ $[\vec{c}_2]_B = \begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}$

e.g. #8 $\vec{b}_1 = \begin{bmatrix} -1 \\ 8 \end{bmatrix}$ $\vec{b}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$

$\vec{c}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ $\vec{c}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Find $\underset{C \leftarrow B}{P}$ and $\underset{B \leftarrow C}{P}$

$\underset{C \leftarrow B}{P} = \left[[\vec{b}_1]_C \ [\vec{b}_2]_C \right]$

$[\vec{b}_1]_C = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ means $\vec{b}_1 = x_1 \vec{c}_1 + x_2 \vec{c}_2$

$$= \begin{bmatrix} \vec{c}_1 & \vec{c}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

solve: $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$[\vec{b}_2]_C = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ means $\vec{b}_2 = y_1 \vec{c}_1 + y_2 \vec{c}_2$

$$= \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

solve $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$

e.g. $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 4 & 1 & 8 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -4 & 3 \end{bmatrix}$

$$\underset{C \leftarrow B}{P} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$$

$$\underset{B \leftarrow C}{P} = \underset{C \leftarrow B}{P}^{-1} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$$ ✓

e.g. $\mathcal{B} = \{1-3t^2, 2+t-5t^2, 1+2t\}$

Spse $\mathcal{B}$ is a basis for $\mathbb{P}_2$.

Find $[t^2]_{\mathcal{B}}$.

Let $\mathcal{E} = \{1, t, t^2\}$ $\quad [t^2]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Need to find $\underset{\mathcal{B} \leftarrow \mathcal{E}}{P} = \Big[[1]_{\mathcal{B}} \; [t]_{\mathcal{B}} \; [t^2]_{\mathcal{B}} \Big]$

Need this

Strategy $\underset{\mathcal{B} \leftarrow \mathcal{E}}{P} = \underset{\mathcal{E} \leftarrow \mathcal{B}}{P}^{-1}$

$= \Big[[1-3t^2]_{\mathcal{E}} \; [2+t-5t^2]_{\mathcal{E}} \; [1+2t]_{\mathcal{E}} \Big]^{-1}$

$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}^{-1}$

Note: $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underset{\mathcal{E} \leftarrow \mathcal{B}}{P} [t^2]_{\mathcal{B}} \iff \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix} [t^2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ -3 & -5 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ $\quad [t^2]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

This means

$$t^2 = 3(1-3t^2) - 2(2+t-5t^2) + (1+2t)$$