3.3 Cramer's Rule, Volume + Linear Transformations

1. Cramer's Rule.

Consider $A\vec{x} = \vec{b}$, A is $n \times n$ matrix, invertible and $\vec{x}, \vec{b} \in \mathbb{R}^n$

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ then $x_i = \dfrac{\det A_i(\vec{b})}{\det A}$, $i = 1, \dots, n$

where $A_i(\vec{b})$ is defined as follows:

$$A = [\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_n]$$

$$A_i(\vec{b}) = [\vec{a}_1 \ \vec{a}_2 \cdots \vec{a}_{i-1} \ \vec{b} \ \vec{a}_{i+1} \cdots \vec{a}_n]$$

i.e. replace i^{th} column of A with $\vec{b}$.

Why? ① Note that $A_i(\vec{b}) = A \underbrace{I_i(\vec{x})}$

identity matrix with i^{th} column replaced by $\vec{x}$

$= A[\vec{e}_1 \ \vec{e}_2 \cdots \vec{e}_{i-1} \ \vec{x} \ \vec{e}_{i+1} \cdots \vec{e}_n]$

$= [A\vec{e}_1 \ A\vec{e}_2 \cdots A\vec{e}_{i-1} \ A\vec{x} \ A\vec{e}_{i+1} \cdots A\vec{e}_n]$

$= [\vec{a}_1 \ \vec{a}_2 \cdots \vec{a}_{i-1} \ \vec{b} \ \vec{a}_{i+1} \cdots \vec{a}_n]$

③ Note $\det I_i(\vec{x}) = x_i$ $\quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$$\therefore \det(A\, I_i(\vec{x})) = \det A_i(\vec{b})$$

$$\det A \cdot \underbrace{\det I_i(\vec{x})} = \det A_i(\vec{b})$$

$$x_i = \frac{\det A_i(\vec{b})}{\det A}$$

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e.g. $\begin{aligned} 4x_1 + x_2 &= 6 \\ 5x_1 + 2x_2 &= 7 \end{aligned} \rightarrow \begin{bmatrix} 4 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$

$A \quad \vec{x} = \vec{b}$

$|A| = \det(A) = 8-5 = 3$

$A_1(\vec{b}) = \begin{bmatrix} 6 & 1 \\ 7 & 2 \end{bmatrix} \quad |A_1(\vec{b})| = 12-7 = 5$

$A_2(\vec{b}) = \begin{bmatrix} 4 & 6 \\ 5 & 7 \end{bmatrix} \quad |A_2(\vec{b})| = 28-30 = -2$

$x_1 = \frac{|A_1(\vec{b})|}{|A|} = \frac{5}{3} \qquad x_2 = \frac{|A_2(\vec{b})|}{|A|} = \frac{-2}{3}$

e.g #6 p209

$$\begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}$$

A

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 3 \end{vmatrix} = (-1)^{1+2} \cdot 1 \cdot \begin{vmatrix} -1 & 2 \\ 3 & 3 \end{vmatrix} + (-1)^{3+2} \cdot 1 \cdot \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix}$$

$$= 9 - 5 = 4$$

$$|A_1(\vec{b})| = \begin{vmatrix} 4 & 1 & 1 \\ 2 & 0 & 2 \\ -2 & 1 & 3 \end{vmatrix} = -\begin{vmatrix} 2 & 2 \\ -2 & 3 \end{vmatrix} - \begin{vmatrix} 4 & 1 \\ 2 & 2 \end{vmatrix} = -10 - 6 = -16$$

$$|A_2(\vec{b})| = \begin{vmatrix} 2 & 4 & 1 \\ -1 & 2 & 2 \\ 3 & -2 & 3 \end{vmatrix} = -\begin{vmatrix} -1 & 2 & 2 \\ 2 & 4 & 1 \\ 3 & -2 & 3 \end{vmatrix} \overset{-3,\,-2}{=} \begin{vmatrix} 1 & -2 & -2 \\ 2 & 4 & 1 \\ 3 & -2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -2 & -2 \\ 0 & 8 & 5 \\ 0 & 4 & 9 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 8 & 5 \\ 4 & 9 \end{vmatrix} = 72 - 20 = 52$$

$$|A_3(\vec{b})| = \begin{vmatrix} 2 & 1 & 4 \\ -1 & 0 & 2 \\ 3 & 1 & -2 \end{vmatrix} = -\begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} - \begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = 4 - 8 = -4$$

$$x_1 = \frac{|A_1(\vec{b})|}{|A|} = \frac{-16}{4} = -4 \qquad x_2 = \frac{|A_2(\vec{b})|}{|A|} = \frac{52}{4} = 13$$

$$x_3 = \frac{|A_3(\vec{b})|}{|A|} = \frac{-4}{4} = -1 \qquad \vec{x} = \begin{bmatrix} -4 \\ 13 \\ -1 \end{bmatrix}$$

2. The adjugate (related to A^{-1}) called adj(A)

Idea: $A = [\vec{a}_1 \cdots \vec{a}_n]$ $\quad \vec{a}_i \in \mathbb{R}^n$

"in" "element of"

Spse A invertible with

$$A^{-1} = [\vec{x}_1 \ \vec{x}_2 \cdots \vec{x}_n]. \text{ Then}$$

$$I = AA^{-1} = A[\vec{x}_1 \ \vec{x}_2 \cdots \vec{x}_n] = [A\vec{x}_1 \ A\vec{x}_2 \cdots A\vec{x}_n]$$
$$= [\vec{e}_1 \ \vec{e}_2 \cdots \vec{e}_n]$$

This means

$$A\vec{x}_1 = \vec{e}_1 \quad A\vec{x}_2 = \vec{e}_2 \quad \cdots \quad A\vec{x}_n = \vec{e}_n$$

$$\left([A \ I] = [A \ \vec{e}_1 \ \vec{e}_2 \cdots \vec{e}_n] \sim [I \ \underbrace{\vec{x}_1 \ \vec{x}_2 \cdots \vec{x}_n}_{A^{-1}}] \right)$$

$\therefore$ j^{th} column of A^{-1}, $\vec{x}_j$, is the solution to $A\vec{x}_j = \vec{e}_j$. So by Cramer:

$$x_{ij} = i^{th} \text{ entry of } \vec{x}_j = \frac{|A_i(\vec{e}_j)|}{|A|}$$

What is $|A_i(\vec{e}_j)|$?

e.g.

$$\left|A_2(\vec{e}_3)\right| = \left|\begin{bmatrix} * & 0 & * & * \\ * & 0 & * & * \\ * & 1 & * & * \\ * & 0 & * & * \end{bmatrix}\right| = (-1)^{3+2}\left|A_{32}\right|$$

A is 4x4

eliminate 3rd row, 2nd column of A.

$$= C_{32} \leftarrow \text{cofactor}$$

$$A^{-1} = \frac{1}{|A|}\underbrace{\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}}_{\text{adj}(A)}$$

e.g. $\begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 3 \end{bmatrix}^{-1}$ $\qquad |A| = +4$

$$C_{11} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} = -2 \cdot (-1)^{1+1} = -2$$

$$C_{12} = \begin{vmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 0 & 3 \end{vmatrix} = (-1)\begin{vmatrix} -1 & 2 \\ 3 & 3 \end{vmatrix} = 9$$

$$C_{13} = \begin{vmatrix} 2 & 1 & 1 \\ -1 & 0 & 0 \\ 3 & 1 & 0 \end{vmatrix} = -1 \qquad C_{21} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} = -2$$

$$C_{22} = \begin{vmatrix} 2 & 0 & 1 \\ -1 & 1 & 2 \\ 3 & 0 & 3 \end{vmatrix} = 3 \qquad C_{23} = \begin{vmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 3 & 1 & 0 \end{vmatrix} = +1$$

$$C_{31} = \begin{vmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 3 \end{vmatrix} = +2 \qquad C_{32} = \begin{vmatrix} 2 & 0 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 3 \end{vmatrix} = -5$$

$$C_{33} = \begin{vmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \\ 3 & 1 & 1 \end{vmatrix} = 1$$

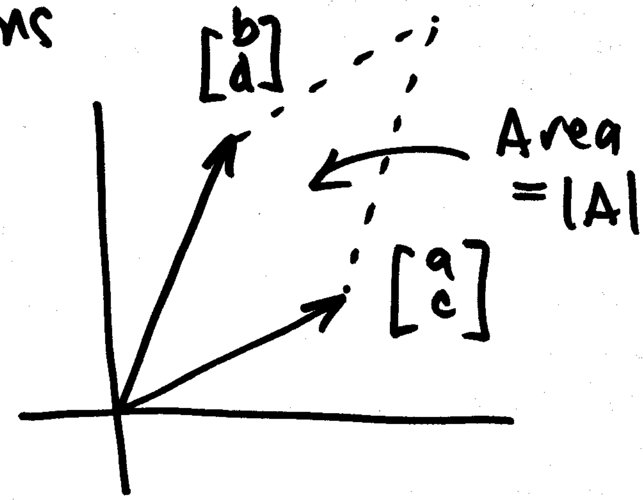
$$A^{-1} = \frac{1}{4}\begin{bmatrix} -2 & -2 & 2 \\ 9 & 3 & -5 \\ -1 & 1 & 1 \end{bmatrix}$$

3. The determinate as a volume (or area)

Idea: In 2-dimensions

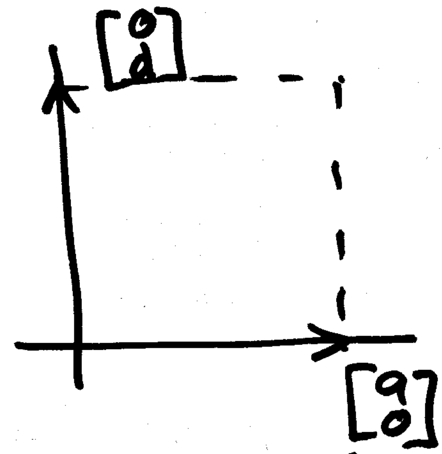
$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$|A| = ad - bc$

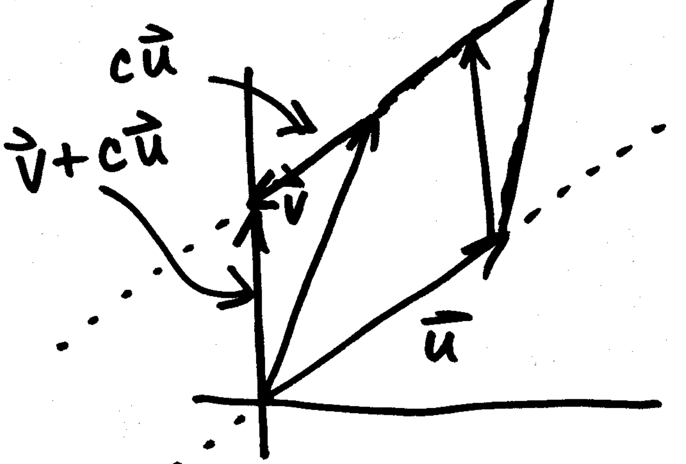


Spse $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$

$|A| = ad = \text{Area}$



Say $A = \begin{bmatrix} \vec{u} \\ \vec{v} \end{bmatrix}$



New parallelogram corresp to:

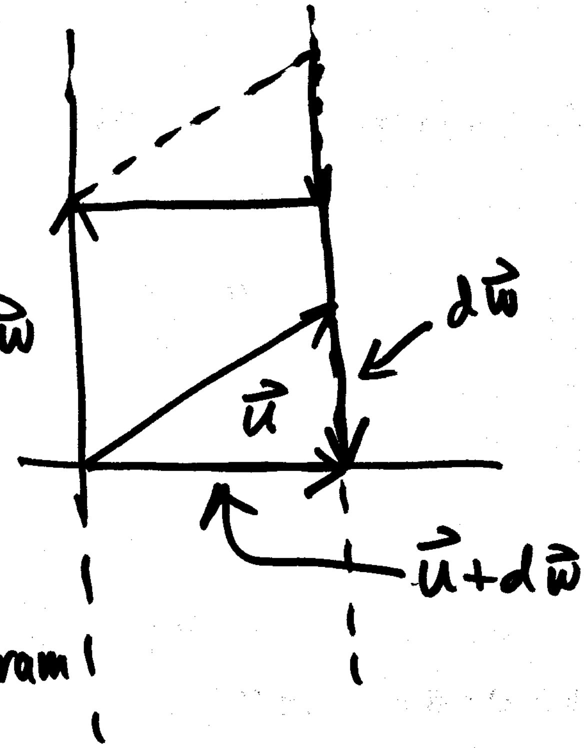
$B = \begin{bmatrix} \vec{u} \\ \vec{v} + c\vec{u} \end{bmatrix}$ ← i.e. B is row equivalent to A

$= \begin{bmatrix} \vec{u} \\ \vec{w} \end{bmatrix}$

New parallelogram corresponds to

$$C = \begin{bmatrix} \vec{u} + d\vec{w} \\ \vec{w} \end{bmatrix}$$

i.e. C is row equiv. to B.



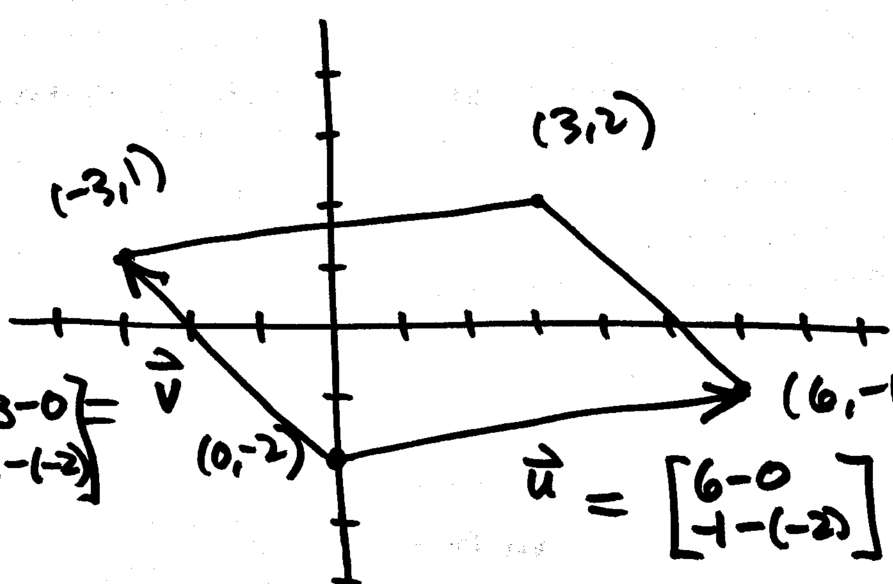
We know: Area of parallelogram

$= |C|$

But since $A \sim B \sim C$ we know $|A| = |B| = |C|$

$\therefore$ Area of original parallelogram is $|A|$.

e.g #22 p210 $(0,-2)$ $(6,-1)$ $(-3,1)$ $(3,2)$



$$\begin{bmatrix} -3 \\ 3 \end{bmatrix} = \begin{bmatrix} -3-0 \\ 1-(-2) \end{bmatrix} = \vec{v}$$

$$\vec{u} = \begin{bmatrix} 6-0 \\ -1-(-2) \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

$$\text{Area} = \left| \begin{matrix} 6 & 1 \\ -3 & 3 \end{matrix} \right| = 21$$