MATH 203 - EXAM 3 - SOLUTIONS

1. Let $\vec{v}_1 = \begin{bmatrix} 3 \\ 6 \\ -9 \\ -3 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -1 \\ -2 \\ 3 \\ 1 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \end{bmatrix}$

Consider $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Since $\vec{v}_2 = -\frac{1}{3}\vec{v}_1$ we can throw out $\vec{v}_2$. Since $\vec{v}_3$ is not a multiple of $\vec{v}_1$, $\{\vec{v}_1, \vec{v}_3\}$ is a linearly independent set. Hence

$\left\{ \begin{bmatrix} 3 \\ 6 \\ -9 \\ -3 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \end{bmatrix} \right\}$ is a basis for S and dim(S) = 2.

2. $$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) rank(A) = dim Col(A) = 2 since A has 2 pivot columns

dim Nul(A) = 3 since rank(A) + dim Nul(A) = 5

(b) Basis for Col(A) = $\left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$

Basis for Row(A) = $\left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ -2 \end{bmatrix} \right\}$

3. Let $\mathcal{E} = \{1, t, t^2\}$ the standard basis for $\mathbb{P}_2$.

Then $[t^2+1]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $[1-t^2]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $[1+t]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

This matrix has a pivot in every row + column. Hence its columns form a basis for $\mathbb{R}^3$. Hence $\mathcal{B}$ forms a basis for $\mathbb{P}_2$. //

4. (a) $\underset{\mathcal{E} \leftarrow \mathcal{B}}{P} = \left[\begin{bmatrix} 1 \\ -2 \end{bmatrix}_{\mathcal{E}} \begin{bmatrix} 5 \\ -6 \end{bmatrix}_{\mathcal{E}} \right] = \begin{bmatrix} 1 & 5 \\ -2 & -6 \end{bmatrix}$

$\underset{\mathcal{E} \leftarrow \mathcal{C}}{P} = \left[\begin{bmatrix} 3 \\ -2 \end{bmatrix}_{\mathcal{E}} \begin{bmatrix} -1 \\ 0 \end{bmatrix}_{\mathcal{E}} \right] = \begin{bmatrix} 3 & -1 \\ -2 & 0 \end{bmatrix}$

(b) $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \left[\begin{bmatrix} 1 \\ -2 \end{bmatrix}_{\mathcal{C}} \begin{bmatrix} 5 \\ -6 \end{bmatrix}_{\mathcal{C}} \right]$

If $\vec{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}_{\mathcal{C}}$ then $\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \underset{\mathcal{E} \leftarrow \mathcal{C}}{P} \vec{x} = \begin{bmatrix} 3 & -1 \\ -2 & 0 \end{bmatrix} \vec{x}$

Solving: $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

If $\vec{y} = \begin{bmatrix} 5 \\ -6 \end{bmatrix}_{\mathcal{C}}$ then $\begin{bmatrix} 5 \\ -6 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 0 \end{bmatrix} \vec{y}$ solving: $\vec{y} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\therefore \underset{C \leftarrow B}{P} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ //

$\underset{B \leftarrow C}{P} = \underset{C \leftarrow B}{P}^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}^{-1} = -\frac{1}{2}\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$ //

5. Eigenspace for $\lambda = 2$ = Nul$(A-2I)$

$$A - 2I = \begin{bmatrix} -2 & -4 & -6 \\ -1 & -2 & -3 \\ 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ -2 & -4 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore (A-2I)\vec{x} = 0 \Rightarrow$

$x_1 = -2x_2 - 3x_3$
x_2 free
x_3 free

$$\vec{x} = \begin{bmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Basis for Eigenspace $= \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ //