MATH 203 - EXAM 4 - SOLUTIONS

1. $H = \{ p(t) = a + (a+b)t + bt^2 : a, b \in \mathbb{R} \}$

$a + (a+b)t + bt^2 = a + at + bt + bt^2$

$= a(1+t) + b(t+t^2)$

$\therefore H = \text{Span}\{(1+t), (t+t^2)\}.$ //

2. $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix} \begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix} \right\}$

$$A = \begin{bmatrix} -1 & 3 & 4 \\ 2 & -5 & -7 \\ 0 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} -1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} -1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Linear independence: Since the matrix A has a pivot in each column, its columns are linearly independent

(b) Spans all of $\mathbb{R}^3$: Since A has a pivot in each <u>row</u>, its columns span all of $\mathbb{R}^3$

$\therefore$ $\mathcal{B}$ is a <u>basis</u> for $\mathbb{R}^3$.

3. $A = \begin{bmatrix} 1 & 0 & -5 & 1 & 4 \\ -2 & 1 & 6 & -1 & -2 \\ 0 & 2 & -8 & 2 & 12 \end{bmatrix}$ A is 3×5

(a) $Nul(A) \leq \mathbb{R}^5$ $\quad Col(A) \leq \mathbb{R}^3$

(b) $Nul(A)$: $\begin{bmatrix} 1 & 0 & -5 & 1 & 4 & 0 \\ -2 & 1 & 6 & -1 & -2 & 0 \\ 0 & 2 & -8 & 2 & 12 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -5 & 1 & 4 & 0 \\ 0 & 1 & -4 & 1 & 6 & 0 \\ 0 & 2 & -8 & 2 & 12 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 1 & 4 & 0 \\ 0 & 1 & -4 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -5 & 1 & 4 & 0 \\ 0 & 1 & -4 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

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$x_1 = 5x_3 - x_4 - 4x_5$
$x_2 = 4x_3 - x_4 - 6x_5$
x_3 free
x_4 = free
x_5 free

$\vec{x} = x_3 \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -4 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ Basis for NulA $= \left\{ \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$Col(A)$: Pivot columns of A are #1, #2

$\therefore$ Basis for $Col(A)$ $= \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ //

4. (a) $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ means $\vec{x} = 3\vec{b}_1 - 2\vec{b}_2$

OR

$$\vec{x} = 3\begin{bmatrix} 1 \\ -2 \end{bmatrix} - 2\begin{bmatrix} 5 \\ -6 \end{bmatrix} = \begin{bmatrix} -7 \\ 6 \end{bmatrix} \quad //$$

(b) $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ means $\begin{bmatrix} 3 \\ -2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ -2 \end{bmatrix} + x_2\begin{bmatrix} 5 \\ -6 \end{bmatrix}$

OR $\begin{bmatrix} 1 & 5 \\ -2 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 5 & 3 \\ -2 & -6 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 3 \\ 0 & 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 3 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad //$$