MATH 203 - EXAM 3 - SOLUTIONS

1. (a) $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 0 & 4 \\ 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & -1 & -2 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -\frac{7}{3} \end{bmatrix}}_{= U}$ //

$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & \frac{1}{3} & 1 \end{bmatrix}$ //

(b) $LU\vec{x} = \vec{b}$ $\quad \underbrace{L\vec{y} = \vec{b}}_{①}$, $\underbrace{U\vec{x} = \vec{y}}_{②}$

① $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$

$y_1 = 1$
$y_2 = -1 - 3y_1 = -4$
$y_3 = 3 - 2y_1 - \frac{1}{3}y_2 = 3 - 2 + \frac{4}{3} = \frac{7}{3}$

② $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -\frac{7}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ \frac{7}{3} \end{bmatrix}$

$x_1 = 1 - x_2 - x_3 = 1 - 1 + 1 = \underline{1}$
$-3x_2 = -4 - x_3 = -3 \quad \underline{x_2 = 1}$
$\underline{x_3 = -1}$

$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ //

(c) $\det(A) = \det(LU) = \det(L)\det(U)$
$= 1 \cdot (1 \cdot -3 \cdot -\frac{7}{3}) = 7$ //

2. $$\begin{vmatrix} 0 & 3 & 1 & 0 \\ 0 & 1 & 5 & 0 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -2 & -8 & -5 \\ 0 & 1 & 5 & 0 \\ 0 & 3 & 1 & 0 \\ 3 & -1 & 2 & 3 \end{vmatrix}$$

EXCH ROWS 1+3
FACTOR −1 FROM ROW 1

$$= \begin{vmatrix} 1 & -2 & -8 & -5 \\ 0 & 1 & 5 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 5 & 26 & 18 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 0 \\ 3 & 1 & 0 \\ 5 & 26 & 18 \end{vmatrix}$$

ADD −3·ROW 1 TO ROW 4

COFACTOR EXP. ON COLUMN 1

$$= 18 \begin{vmatrix} 1 & 5 \\ 3 & 1 \end{vmatrix} = (18)(1-15) = (18)(-14) = -252$$

COFACTOR EXP ON COLUMN 3

COFACTOR EXP ON COLUMN 1

OR $$\begin{vmatrix} 0 & 3 & 1 & 0 \\ 0 & 1 & 5 & 0 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & 2 & 3 \end{vmatrix} = (-1) \begin{vmatrix} 3 & 1 & 0 \\ 1 & 5 & 0 \\ -1 & 2 & 3 \end{vmatrix} - 3 \begin{vmatrix} 3 & 1 & 0 \\ 1 & 5 & 0 \\ 2 & 8 & 5 \end{vmatrix}$$

$$= (-1)(3) \begin{vmatrix} 3 & 1 \\ 1 & 5 \end{vmatrix} - 3(5) \begin{vmatrix} 3 & 1 \\ 1 & 5 \end{vmatrix} = -3(14) - 15(14)$$
$$= -252$$

COFACTOR EXP. ON COLUMN 3 FOR BOTH DETERMINANTS

$\boxed{\text{THERE ARE MANY OTHER SOLUTIONS}}$

3. Check $\begin{vmatrix} 0 & 4 & 5 \\ 1 & 6 & -1 \\ -3 & 6 & -9 \end{vmatrix} = -3 \begin{vmatrix} 0 & 4 & 5 \\ 1 & 6 & -1 \\ 1 & -2 & 3 \end{vmatrix}$

$= -3(-1) \begin{vmatrix} 4 & 5 \\ -2 & 3 \end{vmatrix} - 3(1) \begin{vmatrix} 4 & 5 \\ 6 & -1 \end{vmatrix} = 3\left((12+10) - 3(-4-30)\right)$

$= 3(22+34) = 3(56) = 168 \neq 0$

$\therefore$ Matrix is invertible and columns are <u>linearly independent</u>.

4. $\begin{bmatrix} 3 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$ ① $\begin{vmatrix} 3 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = 3 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -3$ //

② $\begin{vmatrix} -1 & 0 & 4 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = - \begin{vmatrix} -1 & 4 \\ -1 & 1 \end{vmatrix} = -3 \qquad x_1 = \frac{-3}{-3} = 1$ //

③ $\begin{vmatrix} 3 & -1 & 4 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0 \qquad x_2 = 0$ // $\qquad \therefore \vec{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ //

④ $\begin{vmatrix} 3 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{vmatrix} = - \begin{vmatrix} 3 & -1 \\ 0 & -1 \end{vmatrix} = 3 \qquad x_3 = \frac{3}{-3} = -1$ //

5. $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$C_{11} = +\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$ $\quad C_{12} = -\begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} = 0$ $\quad C_{13} = +\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2$

$C_{21} = -\begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} = 3$ $\quad C_{22} = +\begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix} = 0$ $\quad C_{23} = -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1$

$C_{31} = +\begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix} = 0$ $\quad C_{32} = -\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = 6$ $\quad C_{33} = +\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -2$

$\therefore \operatorname{adj}(A) = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 6 \\ 2 & -1 & -2 \end{bmatrix}$ //

$A \cdot \operatorname{adj}(A) = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 6 \\ 2 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 6 \cdot I_3$

$\therefore A^{-1} = \frac{1}{6} \operatorname{adj}(A) = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & -\frac{1}{6} & -\frac{1}{3} \end{bmatrix}$ //