MATH 203 - EXAM 2 - SOLUTIONS

1. $x_1 + 3x_2 = 4$
$x_1 + 4x_2 = 7$

(a) $\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \cdot \frac{1}{(1)(4)-(1)(3)} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} //$$

(b) $$\vec{x} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix} //$$

2. $T(x_1, x_2) = (2x_2 - 3x_1, x_1 - 4x_2, x_2)$

(a) $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3x_1 + 2x_2 \\ x_1 - 4x_2 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} -3 & 2 \\ 1 & -4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad A = \begin{bmatrix} -3 & 2 \\ 1 & -4 \\ 0 & 1 \end{bmatrix}$ //

(b) One-to-one means $A\vec{x} = \vec{b}$ has at most one solution which means the system has no free variables which means A has a pivot in each column:

$$\begin{bmatrix} -3 & 2 \\ 1 & -4 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 \\ 0 & 1 \\ -3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 \\ 0 & 1 \\ 0 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

A has a pivot in each column, so <u>T is one-to-one</u>

(c) Solve $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ $\quad \begin{bmatrix} -3 & 2 & 1 \\ 1 & -4 & 0 \\ 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & -2 \\ -3 & 2 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & -2 \\ 0 & -10 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & -19 \end{bmatrix}$ $\quad$ system is <u>not</u> consistent so $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ is <u>not</u> in Range of T. //

3. (a) $A^T B = \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 2 \end{bmatrix}$ not defined

$3 \times \boxed{2 \quad 3} \times 2$

(b) $(BA)^T = A^T B^T = \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -7 & 5 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(c) $BC = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} -9 & -2 \\ 14 & 4 \\ -9 & -2 \end{bmatrix}$ //

(d) $CB = \begin{bmatrix} 1 & 2 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 2 \end{bmatrix}$ not defined

$2 \times \boxed{2 \quad 3} \times 2$

4. (a) The columns of A, $\left\{ \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ -9 \end{bmatrix} \right\}$, contains the zero vector, hence is not linearly independent. Hence $A\vec{x} = \vec{0}$ has nontrivial solutions hence A is <u>not</u> invertible.

(b) Column 3 of A is a multiple of column 1 so the columns of A do not form a linearly independent set. Hence A is not invertible as above.

5. $$\begin{bmatrix} 1 & 3 & -5 & 1 & 0 & 0 \\ 1 & 4 & -8 & 0 & 1 & 0 \\ -3 & -7 & 10 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1 & 1 & 0 \\ 0 & 2 & -5 & 3 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -5 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 26 & -10 & 5 \\ 0 & 1 & 0 & 14 & -5 & 3 \\ 0 & 0 & 1 & 5 & -2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -16 & 5 & -4 \\ 0 & 1 & 0 & 14 & -5 & 3 \\ 0 & 0 & 1 & 5 & -2 & 1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -16 & 5 & -4 \\ 14 & -5 & 3 \\ 5 & -2 & 1 \end{bmatrix}$$