MATH 203 - EXAM 1 - SOLUTIONS

1. $x_1 + 3x_2 - 5x_3 = 4$
$x_1 + 4x_2 - 8x_3 = 7$
$-3x_1 - 7x_2 + 9x_3 = -6$

(a) $\begin{bmatrix} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{bmatrix}$

(b) $x_1 \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \\ -7 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ -8 \\ 9 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 6 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 & -5 \\ 1 & 4 & -8 \\ -3 & -7 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ -6 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 2 & -6 & 6 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_1 = -5 - 4x_3$
$x_2 = 3 + 3x_3$
x_3 free

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$ //

2. (a) $\begin{bmatrix} -5 & 7 & 9 \\ 1 & -2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 6 \\ -5 & 7 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 6 \\ 0 & -3 & 39 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -2 & 6 \\ 0 & 1 & -13 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -20 \\ 0 & 1 & -13 \end{bmatrix}$

echelon form   reduced echelon form

(b) $x_1 = 20x_3$

$x_2 = 13x_3$

x_3 free

$\vec{x} = x_3 \begin{bmatrix} 20 \\ 13 \\ 1 \end{bmatrix}$

3. (a) $\begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$ cannot be a multiple of $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ because of the zero in the second position. Hence the vectors are linearly independent and span a plane in $\mathbb{R}^3$.

(b) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ -2 & -4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$

Solving $x_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ leads to an equation $0 x_3 = -2$ showing the system to be inconsistent. Hence $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ is <u>not</u> in the span of the columns of A.

4. Reducing A to echelon form:

$$\begin{bmatrix} 0 & 4 & 5 \\ 1 & 6 & -1 \\ -3 & 6 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & -1 \\ 0 & 4 & 5 \\ -3 & 6 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & -1 \\ 0 & 4 & 5 \\ 0 & 24 & -12 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 6 & -1 \\ 0 & 1 & 5/4 \\ 0 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & -1 \\ 0 & 1 & 5/4 \\ 0 & 0 & -7/2 \end{bmatrix}$$

A has a pivot in each row, hence its columns span all of $\mathbb{R}^3$.
Hence the equation $A\vec{x}=\vec{b}$ would never reduce to an equation of the form $0=1$, that is $A\vec{x}=\vec{b}$ is always a consistent system.

5 (a) Reduce matrix to echelon form.

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ 5 & 4 & -1 \\ -4 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ 0 & 4 & -16 \\ 0 & -3 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A does not have a pivot in third column. Hence the equation $A\vec{x} = \vec{0}$ has free variables and non-trivial solutions.

(b) Solving $A\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ 5 & 4 & -1 & 0 \\ -4 & -3 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -3x_3$
$x_2 = 4x_3$
x_3 free

Taking $x_3 = 1$ gives $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$

So

$$-3\begin{bmatrix} 1 \\ 0 \\ 5 \\ -4 \end{bmatrix} + 4\begin{bmatrix} 0 \\ -1 \\ 4 \\ -3 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$