Section 5.2 The Characteristic Equation

Goal: Find eigenvalues of an $n \times n$ matrix.

From Section 2.2,

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

and if $ad - bc = 0$, then $A$ is not invertible.

For a $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the quantity $ad - bc$ is called the determinant of $A$.

We see that

$$\text{det}A = ad - bc.$$  

and

A $2 \times 2$ matrix $A$ is not invertible if and only if $\text{det}A = 0$.

Circle the correct answer: $Ax = 0$ has nontrivial solutions if and only if

(a) $\text{det}A = 0$  
(b) $\text{det}A \neq 0$

EXAMPLE Find the eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$.

Solution: Find all scalars $\lambda$ such that

$$(A - \lambda I)x = 0$$

has a nontrivial solution and this occurs when $\text{det}(A - \lambda I) = 0$. Since

$$A - \lambda I = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ -6 & 5 - \lambda \end{bmatrix},$$

the equation $\text{det}(A - \lambda I) = 0$ becomes

$$-\lambda(5 - \lambda) + 6 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0.$$  

characteristic equation

Now factor:

$$(\lambda - 2)(\lambda - 3) = 0.$$  

So the eigenvalues are 2 and 3.
To summarize, we find the eigenvalues of a $2 \times 2$ matrix $A$ by solving the equation

$$\det(A - \lambda I) = 0$$

for $\lambda$.

How do we compute eigenvalues if $A$ is $3 \times 3$ or higher? To answer this question, we first need to discuss how to compute the determinant of an $n \times n$ matrix.

Suppose the echelon form $U$ is obtained from $A$ by a sequence of row replacements or interchanges, but without scaling.

$$A \sim U = \begin{bmatrix}
    u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\
    0 & u_{22} & u_{23} & \cdots & u_{2n} \\
    0 & 0 & u_{33} & \cdots & u_{3n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & 0 & u_{nn}
\end{bmatrix}$$

The **determinant** of $A$, written $\det A$, is defined as follows:

$$\det A = \begin{cases}
    (-1)^r \cdot \left( \text{product of pivots in } U \right), & \text{when } A \text{ is invertible} \\
    0, & \text{when } A \text{ is not invertible}
\end{cases}$$

At least one of the diagonal elements of $U$ is zero if and only if $\det A = 0$.

At least one of the diagonal elements of $U$ is zero if and only if $A$ is not invertible.

$A$ is invertible if and only if $\det A \neq 0$. 
EXAMPLE  Compute $\det A$ for $A = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 8 & 3 \\ -6 & -8 & -10 \end{bmatrix}$.

Solution:

$$A \sim \begin{bmatrix} 2 & 4 & 6 \\ 0 & 0 & -9 \\ 0 & 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & 6 \\ 0 & 0 & -9 \\ 0 & 4 & 8 \end{bmatrix} = U$$

There was one row interchange and so $r = \ldots$. Therefore, $\det A = \ldots = \ldots$.

The Characteristic Equation

Eigenvalues of $A$ are all scalars $\lambda$ such that $(A - \lambda I)x = 0$ has a nontrivial solution. This occurs when $(A - \lambda I)$ is not invertible or equivalently, when \( \det(A - \lambda I) = 0 \).

A scalar $\lambda$ is an eigenvalue of an $n \times n$ matrix $A$ if and only if $\lambda$ satisfies the characteristic equation

$$\det(A - \lambda I) = 0.$$  

**Characteristic polynomial:** $\det(A - \lambda I)$

**Characteristic equation:** $\det(A - \lambda I) = 0$
EXAMPLE  
Find the characteristic polynomial of

\[
A = \begin{bmatrix}
3 & 2 & 3 \\
0 & 6 & 10 \\
0 & 0 & 2 \\
\end{bmatrix}
\]

and then find all the eigenvalues.

Solution:

\[
\det(A - \lambda I) = \det\begin{bmatrix}
3 - \lambda & 2 & 3 \\
0 & 6 - \lambda & 10 \\
0 & 0 & 2 - \lambda \\
\end{bmatrix}
\]

Characteristic equation: \((\quad)(\quad)(\quad) = 0\).

Now we solve this system to find the eigenvalues:

**eigenvalues:** _____, _____, _____

Chapter Three Results

**THEOREM 3  PROPERTIES OF DETERMINANTS**

Let \(A\) and \(B\) be \(n \times n\) matrices.

a. \(A\) is invertible if and only if \(\det A \neq 0\).

b. \(\det AB = (\det A)(\det B)\)

c. \(\det A^T = \det A\)

d. If \(A\) is triangular, then \(\det A\) is the product of the entries on the main diagonal of \(A\).

e. A row replacement operation on \(A\) does not change the determinant. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same scaling factor.

In general, the (algebraic) multiplicity of an eigenvalue is its multiplicity as a root of the characteristic equation.
EXAMPLE: Find the characteristic polynomial of \( A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 5 & 3 & 0 & 0 \\ 9 & 1 & 3 & 0 \\ 1 & 2 & 5 & -1 \end{bmatrix} \)

and then find all the eigenvalues and the algebraic multiplicity of each eigenvalue.

Solution: \( A - \lambda I = \begin{bmatrix} 2 - \lambda & 0 & 0 & 0 \\ 5 & 3 - \lambda & 0 & 0 \\ 9 & 1 & 3 - \lambda & 0 \\ 1 & 2 & 5 & -1 - \lambda \end{bmatrix} \)

Since \( \det(A - \lambda I) = \det((A - \lambda I)^T) \),

\[
\det(A - \lambda I) = \det \begin{bmatrix} 2 - \lambda & 5 & 9 & 1 \\ 0 & 3 - \lambda & 1 & 2 \\ 0 & 0 & 3 - \lambda & 5 \\ 0 & 0 & 0 & -1 - \lambda \end{bmatrix} = (2 - \lambda)(3 - \lambda)(3 - \lambda)(-1 - \lambda)
\]

Characteristic Polynomial:

\( (2 - \lambda)(3 - \lambda)(3 - \lambda)(-1 - \lambda) \)

Characteristic Equation:

\( (2 - \lambda)(3 - \lambda)(3 - \lambda)(-1 - \lambda) = 0 \)

eigenvalues: \( _____, _____, _____ \)

Similarity

Numerical methods for finding approximating eigenvalues are based upon Theorem 4 to be described shortly. For \( n \times n \) matrices \( A \) and \( B \), we say the \( A \) is similar to \( B \) if there is an invertible matrix \( P \) such that

\[ P^{-1}AP = B \quad \text{or equivalently,} \quad A = PBP^{-1}. \]
EXAMPLE: Show that \((\det P)(\det P^{-1}) = 1\) if \(P\) is invertible and then show that \(\det A = \det B\) if \(A\) is similar to \(B\).

Solution: By Theorem 3(b),

\[(\det P)(\det P^{-1}) = \det(PP^{-1})\]

and therefore

\[(\det P)(\det P^{-1}) = \det(I) = 1.\]

If \(A\) is similar to \(B\), then

\[A = PBP^{-1}\]

\[\det(A) = \det(PBP^{-1}) = (\underline{\text{\hspace{1cm}}})(\underline{\text{\hspace{1cm}}})(\underline{\text{\hspace{1cm}}})\]

\[= (\underline{\text{\hspace{1cm}}})(\underline{\text{\hspace{1cm}}})(\underline{\text{\hspace{1cm}}})\]

\[= (\underline{\text{\hspace{1cm}}})(\underline{\text{\hspace{1cm}}}) = \underline{\text{\hspace{1cm}}}\]

Theorem 4: If \(n \times n\) matrices \(A\) and \(B\) are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

Proof: If \(B = P^{-1}AP\), then

\[\det(B - \lambda I) = \det[P^{-1}AP - P^{-1}\lambda IP] = \det[P^{-1}(A - \lambda I)P] = \det P^{-1} \cdot \det(A - \lambda I) \cdot \det P = \det(A - \lambda I).\]

Application to Markov Chains

EXAMPLE Consider the migration matrix \(M = \begin{bmatrix} .95 & .90 \\ .05 & .10 \end{bmatrix}\) and define \(x_k = M^k x_0\) where

\[x_0 = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}.\]
We have the following eigenvalues and eigenvectors:

<table>
<thead>
<tr>
<th>Eigenvalue</th>
<th>$\lambda = 1$</th>
<th>$\lambda = 0.05$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corresponding Eigenvector</td>
<td>$v_1 = \begin{bmatrix} \frac{18}{19} \ \frac{1}{19} \end{bmatrix}$</td>
<td>$v_2 = \begin{bmatrix} -\frac{1}{2} \ \frac{1}{2} \end{bmatrix}$</td>
</tr>
</tbody>
</table>

(a) Use Theorem 2 (page 301) to explain why $\langle v_1, v_2 \rangle$ is a basis for $\mathbb{R}^2$.
(b) Write $x_0$ as a linear combination of $v_1$ and $v_2$.
(c) Show that $x_k \to v_1$ as $k$ increases.

**Solution:**

(a)

(b) $c_1v_1 + c_2v_2 = x_0$ corresponds to

$$c_1 \begin{bmatrix} \frac{18}{19} \\ \frac{1}{19} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{4}{5} \end{bmatrix}.

\begin{bmatrix} \frac{18}{19} & -\frac{1}{2} \\ \frac{1}{19} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ \frac{142}{95} \end{bmatrix}

$$x_0 = v_1 + \frac{142}{95}v_2$$

$$x_k = M^kx_0 = M^k\left( v_1 + \frac{142}{95}v_2 \right) = M^kv_1 + \frac{142}{95}M^kv_2$$

Recall that if $Mx = \lambda x$, then $M^kx = \lambda^kx$.

$$x_k = \left( \begin{bmatrix} \frac{18}{19} \\ \frac{1}{19} \end{bmatrix} \right)^k + \frac{142}{95}\left( \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right)^k$$

As $k \to \infty$, $(0.05)^k$ tends to 0

and so $x_k$ tends to $v_1$. 

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