MATH 114 - EXAM 4 - SOLUTIONS

1. (a) $$\sum_{n=0}^{\infty}(-1)^n\frac{2^{n+1}}{3^n} = \sum_{n=0}^{\infty}(-1)^n\frac{2\cdot 2^n}{3^n} = \sum_{n=0}^{\infty}2\cdot\left(\frac{-2}{3}\right)^n$$

$$= \frac{2}{1-\left(-\frac{2}{3}\right)} = \frac{2}{\frac{5}{3}} = 2\cdot\frac{3}{5} = \frac{6}{5}$$ //

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n+1}$ diverges since $\displaystyle\lim_{n\to\infty}\frac{n}{n+1}=1\neq 0.$

(c) $\displaystyle\sum_{n=4}^{\infty}\frac{1}{\sqrt{n+2}}-\frac{1}{\sqrt{n+1}}$ telescoping series.

$$S_n = \cancel{\frac{1}{\sqrt{6}}}-\frac{1}{\sqrt{5}}+\cancel{\frac{1}{\sqrt{7}}}-\cancel{\frac{1}{\sqrt{6}}}+\cancel{\frac{1}{\sqrt{8}}}-\cancel{\frac{1}{\sqrt{7}}}+\cdots+\cancel{\frac{1}{\sqrt{n+2}}}-\frac{1}{\sqrt{n+1}}$$

$$= -\frac{1}{\sqrt{5}}-\frac{1}{\sqrt{n+1}} \longrightarrow -\frac{1}{\sqrt{5}}$$

2. (a) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^{3/2}}$ converges with $\int_1^\infty \frac{\ln x}{x^{3/2}}\,dx$ //

$$\int_1^\infty \frac{\ln x}{x^{3/2}}\,dx \qquad u = \ln x \quad dv = x^{-3/2}dx$$
$$du = \frac{1}{x}dx \quad v = -2x^{-1/2}$$

$$= \lim_{b\to\infty} \left(-2x^{-1/2}\ln x \Big|_1^b + 2\int_1^b x^{-3/2}dx \right)$$

$$= \lim_{b\to\infty} -2\frac{\ln b}{b^{1/2}} - 4x^{-1/2}\Big|_1^b = \lim_{b\to\infty} -2\frac{\ln b}{b^{1/2}} - \frac{4}{\sqrt{b}} + 4$$

$$= 4 < \infty$$

(b) $\sum_{n=1}^{\infty} \frac{1}{2n+1}$ diverges with $\int_1^\infty \frac{1}{2x+1}\,dx$. //

$$\int_1^\infty \frac{1}{2x+1}\,dx = \lim_{b\to\infty} \frac{1}{2}\ln(2x+1)\Big|_1^b$$

$$= \lim_{b\to\infty} \frac{1}{2}\ln(2b+1) - \frac{1}{2}\ln 3 = \infty$$

3. (a) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ Compare with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

Limit comparison test

$$\lim_{n\to\infty} \frac{\frac{1}{n^{3/2}}}{\frac{\sqrt{n}}{n^2+1}} = \lim_{n\to\infty} \frac{n^2+1}{n^{3/2}\sqrt{n}} = \lim_{n\to\infty} \frac{n^2+1}{n^2} = 1$$

$\therefore \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ converges with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, a convergent p-series ($p = 3/2 > 1$).

(b) $\sum_{n=2}^{\infty} \frac{\cos^2(n)}{n^3}$ Compare with $\sum_{n=2}^{\infty} \frac{1}{n^3}$

Direct comparison test

$\frac{\cos^2(n)}{n^3} \le \frac{1}{n^3}$ so $\sum_{n=2}^{\infty} \frac{\cos^2(n)}{n^3}$ converges

with $\sum_{n=2}^{\infty} \frac{1}{n^3}$ a convergent p-series

$p = 3 > 1$.

4. (a) $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$ $\quad \frac{a_{n+1}}{a_n} = \frac{(n+1)^{10}}{10^{n+1}} \cdot \frac{10^n}{n^{10}}$

$$= \left(\frac{n+1}{n}\right)^{10} \cdot \frac{1}{10} \longrightarrow \frac{1}{10} < 1 \text{ as } n \to \infty$$

$\therefore$ Series converges by Ratio test.

(b) $\sum_{n=1}^{\infty} \frac{n^{10} 10^n}{n!}$ $\quad \frac{a_{n+1}}{a_n} = \frac{(n+1)^{10} 10^{n+1}}{(n+1)!} \cdot \frac{n!}{n^{10} 10^n}$

$$= \frac{(n+1)^{10}}{n^{10}} \cdot \frac{10^{n+1}}{10^n} \cdot \frac{n!}{(n+1)!} = \left(\frac{n+1}{n}\right)^{10} \cdot 10 \cdot \frac{1}{n+1}$$

$$\longrightarrow 0 < 1 \quad \text{as } n \to \infty$$

$\therefore$ Series converges by Ratio test.

5. (a) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^{3/2}+n^{1/2}}$

$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1}{n^{3/2}+n^{1/2}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}+n^{1/2}}$ compare to $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

Limit comparison: $\lim_{n\to\infty} \frac{\frac{1}{n^{3/2}}}{\frac{1}{n^{3/2}+n^{1/2}}} = \lim_{n\to\infty} \frac{n^{3/2}+n^{1/2}}{n^{3/2}}$

$= \lim_{n\to\infty} 1 + \frac{1}{n} = 1$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^{3/2}+n^{1/2}}$ converges and $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^{3/2}+n^{1/2}}$ converges absolutely. //

(b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$ $\quad \frac{n}{n^2+1} \geq 0$ all n, $\frac{n}{n^2+1} \to 0$

eventually decreasing

$\therefore$ converges by A.S.T.

$\frac{n}{n^2+1} \geq \frac{n+1}{(n+1)^2+1} \Leftrightarrow n(n+1)^2+n \geq (n+1)(n^2+1)$

$\Leftrightarrow n^3+2n^2+2n \geq n^3+n^2+n+1 \Leftrightarrow$

$n^2+n-1 \geq 0$ TRUE FOR $n \geq 1$

OR $\frac{d}{dx}\left(\frac{x}{x^2+1}\right) = \frac{-x^2+1}{(x^2+1)^2} \leq 0$ for $x \geq 1$

$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{n}{n^2+1} \right| = \sum_{n=1}^{\infty} \frac{n}{n^2+1}$

compare to $\sum_{n=1}^{\infty} \frac{1}{n}$ (divergent p-series)

$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{n}{n^2+1}} = \lim \frac{n^2+1}{n^2} = 1$

$\therefore \sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges with $\sum_{n=1}^{\infty} \frac{1}{n}$ $\therefore$ converges conditionally //