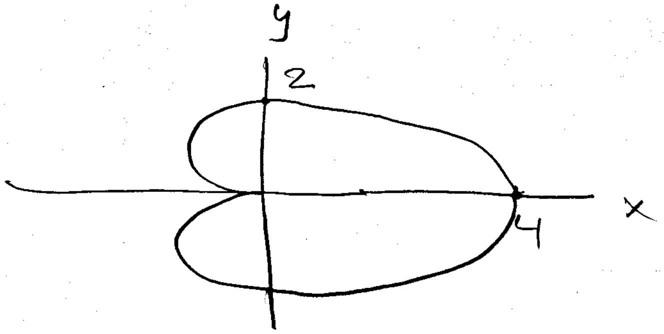
MATH 114 - EXAM 3 - SOLUTIONS

1.



(a) $$A = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2}\left(2(1+\cos\theta)\right)^2 d\theta$$

$$= 2\int_0^{2\pi} (1+\cos\theta)^2 \, d\theta \; //$$

(b) $$L = \int_0^{2\pi} \left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right)^{1/2} d\theta = \int_0^{2\pi} \left(\left(2(1+\cos\theta)\right)^2 + (-2\sin\theta)^2\right)^{1/2} d\theta$$

$$\boxed{\frac{dr}{d\theta} = -2\sin\theta} \quad = \int_0^{2\pi} \left(4(1+\cos\theta)^2 + 4\sin^2\theta\right)^{1/2} d\theta$$

$$= 2\int_0^{2\pi} (2+2\cos\theta)^{1/2} \, d\theta \; //$$

(c) $$x = 2(1+\cos\theta)\cos\theta = 2(\cos\theta + \cos^2\theta)$$

$$y = 2(1+\cos\theta)\sin\theta = 2(\sin\theta + \cos\theta\sin\theta)$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cancel{2}(\cos\theta + \cos^2\theta - \sin^2\theta)}{\cancel{2}(-\sin\theta - 2\cos\theta\sin\theta)}$$

$$\left.\frac{dy}{dx}\right|_{\frac{\pi}{4}=\theta} = \frac{\frac{\sqrt{2}}{2} + \frac{1}{2} - \frac{1}{2}}{-\frac{\sqrt{2}}{2} - 2\cdot\frac{1}{2}} = \frac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2} - 1} = -\frac{\sqrt{2}}{2+\sqrt{2}} \; //$$

2. (a) $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x^4+1}}$ compare with $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$:

$$\lim_{x\to\infty} \frac{\frac{1}{x^2}}{\frac{1}{\sqrt{x^4+1}}} = \lim_{x\to\infty} \frac{\sqrt{x^4+1}}{x^2} = \lim_{x\to\infty} \sqrt{\frac{x^4+1}{x^4}} = 1$$

$\therefore \displaystyle\int_0^\infty \frac{dx}{\sqrt{x^4+1}}$ converges with $\displaystyle\int_1^\infty \frac{dx}{x^2}$ by p-test, $p>1$

(b) $\displaystyle\int_3^\infty \frac{2u^2}{3u^3+1}\,du$ compare with $\displaystyle\int_3^\infty \frac{du}{u}$:

$$\lim_{u\to\infty} \frac{\frac{1}{u}}{\frac{2u^2}{3u^3+1}} = \lim_{u\to\infty} \frac{3u^3+1}{2u^3} = \frac{3}{2} > 0$$

$\therefore \displaystyle\int_3^\infty \frac{2u^2}{3u^3+1}\,du$ diverges with $\displaystyle\int_3^\infty \frac{du}{u}$.

3. (a) $\displaystyle\int_1^\infty \frac{\ln x}{x^3}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^3}\,dx$ $\quad u = \ln x \quad dv = x^{-3}dx$ $\quad du = \frac{1}{x}dx \quad v = -\frac{1}{2}x^{-2}$

$$= \lim_{b\to\infty} \left[-\frac{\ln x}{2x^2}\Big|_1^b + \frac{1}{2}\int_1^b x^{-3}\,dx \right]$$

$$= \lim_{b\to\infty} -\frac{\ln b}{2b^2} - \frac{1}{4}x^{-2}\Big|_1^b = \lim_{b\to\infty} -\frac{\ln b}{2b^2} - \frac{1}{4b^2} + \frac{1}{4} = \frac{1}{4} \,//$$

(b) $\displaystyle\int_0^2 \frac{dx}{(2-x)^{1/2}} = \lim_{c\to 2^-} \int_0^c (2-x)^{-1/2}\,dx = \lim_{c\to 2^-} -2(2-x)^{1/2}\Big|_0^c$

$$= \lim_{c\to 2^-} -2(2-c)^{1/2} + 2\sqrt{2} = 2\sqrt{2} \,//$$

4. (a) $\displaystyle\int_0^1 \frac{dx}{(x^2+1)^2}$

$x = \tan\theta$ $\quad$ $x=0 \quad \theta = \tan^{-1}0 = 0$

$dx = \sec^2\theta\, d\theta$ $\quad$ $x=1 \quad \theta = \tan^{-1}1 = \frac{\pi}{4}$

$(x^2+1)^2 = (\tan^2\theta + 1)^2 = (\sec^2\theta)^2 = \sec^4\theta$

$$= \int_0^{\pi/4} \frac{\sec^2\theta}{\sec^4\theta}\, d\theta = \int_0^{\pi/4} \frac{1}{\sec^2\theta}\, d\theta = \int_0^{\pi/4} \cos^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2} + \frac{1}{2}\cos 2\theta\, d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta \Big|_0^{\pi/4}$$

$$= \frac{1}{2}\cdot\frac{\pi}{4} + \frac{1}{4}\sin\frac{\pi}{2} = \frac{\pi}{8} + \frac{1}{4} \quad //$$

(b) $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\, dx$

$x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4\cos^2\theta} = 2\cos\theta$

$$= \int \frac{4\sin^2\theta}{2\cos\theta}\cdot 2\cos\theta\, d\theta = 4\int \sin^2\theta\, d\theta = 4\int \frac{1}{2} - \frac{1}{2}\cos 2\theta\, d\theta$$

$$= \int 2 - 2\cos 2\theta\, d\theta$$

$$= 2\theta - \sin 2\theta + C$$

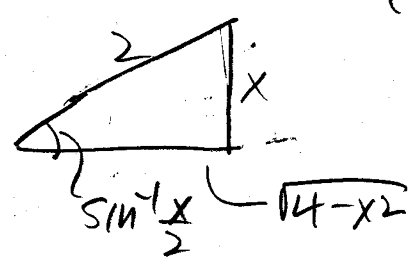
$$= 2\sin^{-1}\left(\frac{x}{2}\right) - \sin 2\left(\sin^{-1}\frac{x}{2}\right) + C$$

$$= 2\sin^{-1}\frac{x}{2} - \frac{x}{2}\sqrt{4-x^2} + C$$

$\theta = \sin^{-1}\left(\frac{x}{2}\right)$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\sin 2\left(\sin^{-1}\frac{x}{2}\right) = 2\sin\left(\sin^{-1}\frac{x}{2}\right)\cdot\cos\left(\sin^{-1}\frac{x}{2}\right)$



$= 2\cdot\frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2}$

5. (a) $\int \frac{x}{(2x+1)^{3/2}}\,dx = \int x(2x+1)^{-3/2}\,dx$ #1 $a=2$ $b=1$

$$= \frac{(2x+1)^{-1/2}}{4}\left[\frac{2x+1}{\frac{1}{2}} - \frac{1}{-\frac{1}{2}}\right] + c$$

$$= \frac{(2x+1)^{1/2}}{2} + \frac{(2x+1)^{-1/2}}{2} + c \quad //$$

(b) $\int x^2(1+9x^2)^{-1/2}\,dx = \int \frac{x^2}{\sqrt{1+9x^2}}\,dx$

$$= \int \frac{x^2}{\sqrt{9(\frac{1}{9}+x^2)}}\,dx = \frac{1}{3}\int \frac{x^2}{\sqrt{\frac{1}{9}+x^2}}\,dx \quad \#6 \quad a=\frac{1}{3}$$

$$= \frac{1}{3}\left[-\frac{1}{18}\ln\left(x+\sqrt{\tfrac{1}{9}+x^2}\right) - \frac{x\sqrt{\frac{1}{9}+x^2}}{2}\right] + c$$

$$= -\frac{1}{54}\ln\left(x+\sqrt{\tfrac{1}{9}+x^2}\right) - \frac{x\sqrt{\frac{1}{9}+x^2}}{6} + c \quad //$$