Final Exam Tuesday 7-31

omitted sections: Chapter 7, 8.6, Chapter 10, 11.1, 8.7. Bring 4 3x5 cards or 1 $8\frac{1}{2}$"x11" sheet

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11.7 7) $\displaystyle\sum_{n=0}^{\infty} \frac{n x^n}{n+2}$

Ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)x^{n+1}}{n+3} \cdot \frac{n+2}{n x^n}\right|$$

$$= |x| \frac{(n+1)(n+2)}{(n+3)\cdot n} \longrightarrow |x|$$

as $n \to \infty$

Interval of convergence:

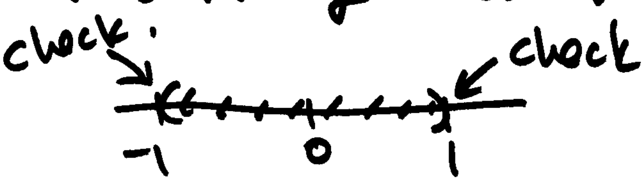
① Always include interval of the form $(a-R, a+R)$

a – center of power series

R – radius of convergence

② May include one or both or neither endpoint.

Converges absolutely when $|x|<1$, i.e. for x in $(-1,1)$.



$\underline{x=-1}$ $\displaystyle\sum_{n=0}^{\infty} \frac{n(-1)^n}{n+2}$ diverges because terms do not go to zero.

$\underline{x=1}$ $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n+2}$ diverges for same reason.

Interval of convergence: $(-1,1)$, $-1<x<1$

15) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2+3}}$

<u>Ratio test:</u>

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{\sqrt{(n+1)^2+3}} \cdot \frac{\sqrt{n^2+3}}{x^n}\right|$$

$$= |x| \frac{\sqrt{n^2+3}}{\sqrt{(n+1)^2+3}} = |x| \sqrt{\frac{n^2+3}{(n+1)^2+3}}$$

$$\longrightarrow |x| \text{ as } n \to \infty.$$

Conv. abs for $|x|<1$, i.e. $(-1,1)$.

Check endpts: <u>$x=-1$</u>

$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2+3}}$ <u>converges</u> by A.S.T.

since $\frac{1}{\sqrt{n^2+3}}$ ~~decreases~~ decreases to zero

as $n \to \infty$.

<u>$x=1$</u> $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+3}}$ Compare with $\frac{1}{n}$.

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{\sqrt{n^2+3}}} = \lim_{n\to\infty} \frac{\sqrt{n^2+3}}{n} = \lim_{n\to\infty} \sqrt{\frac{n^2+3}{n^2}} = 1$$

$\therefore \displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+3}}$ <u>diverges</u> with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ by limit comp test.

Direct comparison:

Want $\frac{1}{n} \leq \frac{1}{\sqrt{n^2+3}}$ NOT TRUE

$\frac{1}{n+3} \leq \frac{1}{\sqrt{n^2+3}} \longleftrightarrow n+3 \geq \sqrt{n^2+3}$

$\longleftrightarrow n^2+6n+9 \geq n^2+3$

$6n \geq -6$

$n \geq -1$ TRUE

So $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+3}}$ diverges with $\sum_{n=0}^{\infty} \frac{1}{n+3}$

OR Try $\frac{1}{2n} \leq \frac{1}{\sqrt{n^2+3}}$ TRUE

$\longleftrightarrow 2n \geq \sqrt{n^2+3} \longleftrightarrow 4n^2 \geq n^2+3 \longleftrightarrow 3n^2 \geq 3$

$\longleftrightarrow n^2 \geq 1$ TRUE

Clearly $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so so does $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+3}}$.

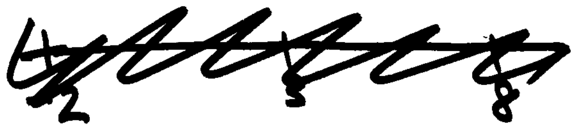
Interval of convergence: $[-1, 1)$, or $-1 \leq x < 1$.

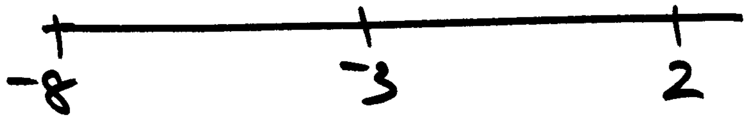
17) $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$ Ratio: $\displaystyle\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)(x+3)^{n+1}}{5^{n+1}} \cdot \frac{5^n}{n(x+3)^n}\right|$

$$= |x+3|\left(\frac{n+1}{n}\right)\cdot\frac{1}{5} \xrightarrow{n\to\infty} \frac{|x+3|}{5}$$

Conv. abs

$\dfrac{|x+3|}{5} < 1$ or $|x+3| < 5$





$(-8, 2)$

$\underline{x=-8}$ $\displaystyle\sum_{n=0}^{\infty} \frac{n(-8+3)^n}{5^n} = \sum_{n=0}^{\infty} \frac{n(-5)^n}{5^n} = \sum_{n=0}^{\infty} n(-1)^n$

diverges since terms do not go to zero.

$\underline{x=2}$ $\displaystyle\sum_{n=0}^{\infty} \frac{n(2+3)^n}{5^n} = \sum_{n=0}^{\infty} n\cdot\frac{5^n}{5^n} = \sum_{n=0}^{\infty} n$

diverges same reason.

Interval of convergence: $(-8, 2)$

37) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{x^2+1}{3}\right)^n$

Interval of convergence:

$$\left|\frac{x^2+1}{3}\right| = \frac{x^2+1}{3} < 1$$

$$x^2+1 < 3$$

$$\sqrt{x^2} < \sqrt{2}$$

$$|x| < \sqrt{2}$$

Abs conv on $(-\sqrt{2}, \sqrt{2})$

Check endpts:

$x = -\sqrt{2}$ $\quad \frac{x^2+1}{3} = \frac{2+1}{3} = 1$

$$\sum_{n=0}^{\infty}\left(\frac{x^2+1}{3}\right)^n = \sum_{n=0}^{\infty} 1^n$$

$$= \sum_{n=0}^{\infty} 1 \text{ diverges.}$$

$x = \sqrt{2}$ $\quad \frac{x^2+1}{3} = \frac{2+1}{3} = 1$

$$\sum_{n=0}^{\infty}\left(\frac{x^2+1}{3}\right)^n = \sum_{n=0}^{\infty} 1 \text{ diverges}$$

$\boxed{\text{Int of conv is } (-\sqrt{2}, \sqrt{2}).}$

$\displaystyle\sum_{n=0}^{\infty} r^n$ Root test

$$(|a_n|)^{1/n} = (|r^n|)^{1/n} = |r| \xrightarrow[n\to\infty]{} |r| < 1$$

Sum the series:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \qquad \sum_{n=0}^{\infty}\left(\frac{x^2+1}{3}\right)^n = \frac{1}{1-\frac{x^2+1}{3}}\cdot\frac{3}{3}$$

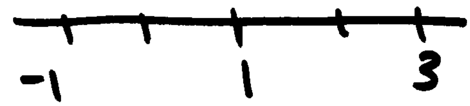
$$= \frac{3}{3-(x^2+1)} = \frac{3}{2-x^2} \; //$$

33) $$\sum_{n=0}^{\infty} \frac{(x-1)^{2n}}{4^n} = \sum_{n=0}^{\infty} \left[\frac{(x-1)^2}{4}\right]^n$$

Abs conv: $$\left|\frac{(x-1)^2}{4}\right| = \frac{(x-1)^2}{4} < 1$$

$$\sqrt{(x-1)^2} < \sqrt{4}$$

$$|x-1| < 2$$



<u>or</u> $(-1, 3)$

Check end points: diverges. $\boxed{\text{Int of conv is } (-1,3).}$

$$\sum_{n=0}^{\infty} \left[\frac{(x-1)^2}{4}\right]^n = \frac{1}{1-\frac{(x-1)^2}{4}} = \frac{4}{4-(x-1)^2}$$

$$= \frac{4}{3+2x-x^2} \quad //$$

11.8 13) $\sin(3x)$

$$\sin(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n+1}$$

$$\sin(3x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (3x)^{2n+1}$$

$$= \sum_{n=0}^{\infty} \underbrace{\frac{(-1)^n}{(2n+1)!} 3^{2n+1}} x^{2n+1}$$

11.9 3) $5\sin(-x) = -5\sin(x)$

$$= \sum_{n=0}^{\infty} (-5) \frac{(-1)^n}{(2n+1)!} x^{2n+1} = 5 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} x^{2n+1}$$

7) xe^x

$$e^t = \sum_{n=0}^{\infty} \frac{1}{n!} t^n$$

$$xe^x = x \cdot \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^{n+1}$$

11) $x\cos(\pi x)$

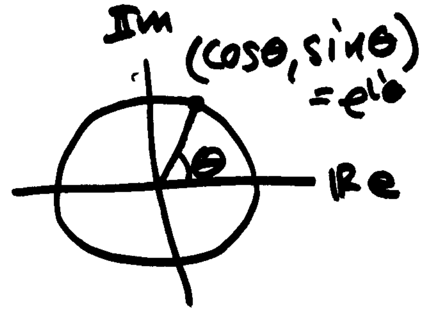
$$\boxed{\cos t = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n}}$$

$$= x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\pi x)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!} x^{2n+1}$$

49) $e^{i\theta} = \cos\theta + i\sin\theta \qquad i^2 = -1$

$$e^{-i\pi} = e^{i(-\pi)} = \cos(-\pi) + i\sin(-\pi)$$

$$= -1 + i\cdot(0) = -1.$$



$$e^{i\frac{\pi}{4}} = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$$

$$= \frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}.$$

11.10 7) $(1+x^3)^{-1/2}$

$$-\frac{\overset{5}{\cancel{15}}}{8}\cdot\frac{1}{\cancel{6}\,2}$$

Know $(1+t)^{-1/2} = \sum_{k=0}^{\infty} \binom{-1/2}{k} t^k$

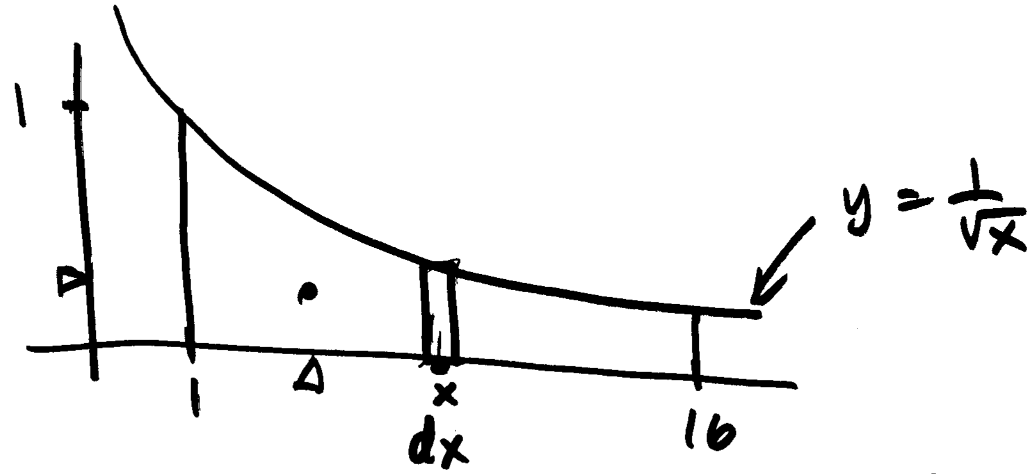
$$= 1 + \left(-\tfrac{1}{2}\right)t + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}t^2 + \frac{\overbrace{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}^{-\frac{15}{8}}}{\underbrace{3!}_{6}}t^3 + \cdots$$

$$\boxed{\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}}$$

$$= 1 - \frac{1}{2}t + \frac{3}{8}t^2 - \frac{5}{16}t^3 + \cdots$$

$$(1+x^3)^{-1/2} = 1 - \frac{1}{2}x^3 + \frac{3}{8}x^6 - \frac{5}{16}x^9 + \cdots$$

6.4 29) $y = \frac{1}{\sqrt{x}}$ $x=1$, $x=16$, x-axis



$$\delta(x) = \frac{4}{\sqrt{x}}$$

$$M_x = \int_1^{16} dM_x = \int_1^{16} \begin{bmatrix}\text{mass}\\ \text{of}\\ \text{rect.}\end{bmatrix} \begin{bmatrix}\text{"distance"}\\ \text{from } x\text{-axis}\end{bmatrix}$$

$$\begin{bmatrix}\text{area of}\\ \text{rect}\end{bmatrix} \times \begin{bmatrix}\text{density}\end{bmatrix}$$

$$= \int_1^{16} \left(\frac{1}{\sqrt{x}} \cdot \frac{\cancel{4}^{\,2}}{\sqrt{x}} \frac{1}{\cancel{2}} \cdot \frac{1}{\sqrt{x}}\right) dx$$

$$= \int_1^{16} \frac{2}{x^{3/2}}\, dx = \int_1^{16} 2x^{-3/2}\, dx$$

$$= 2(-2)x^{-1/2}\Big|_1^{16} = -4\left(\frac{1}{4} - 1\right) = -4 \cdot -\frac{3}{4} = 3$$

$$M_y = \int_1^{16} dM_y = \int_1^{16} \begin{bmatrix}\text{mass of}\\ \text{rectangle}\end{bmatrix} \begin{bmatrix}\text{"distance"}\\ \text{from y-axis}\end{bmatrix}$$

$$= \int_1^{16} \frac{1}{\sqrt{x}} \cdot \frac{4}{\sqrt{x}} \cdot x \quad dx = \int_1^{16} 4 \, dx = 60$$

$$M = \int_1^{16} dM = \int_1^{16} \begin{bmatrix}\text{mass of}\\ \text{rectangle}\end{bmatrix}$$

$$= \int_1^{16} \frac{1}{\sqrt{x}} \cdot \frac{4}{\sqrt{x}} \, dx = \int_1^{16} \frac{4}{x} dx$$

$$= 4 \ln x \Big|_1^{16} = 4 \ln(16) - 4 \ln(1) = 4 \ln 16$$

$$= 4 \ln(2^4) = 16 \ln 2$$

$$\bar{x} = \frac{M_y}{M}$$

$$= \frac{60}{16 \ln 2} \approx 5.4$$

$$\bar{y} = \frac{M_x}{M} = \frac{3}{16 \ln 2} \approx .27$$

8.4 11) $\int_0^{\frac{\pi}{2}} 35 \sin^4 x \cos^3 x \, dx$

$= \int_0^{\frac{\pi}{2}} 35 \sin^4 x \cos^2 x \cos x \, dx$

$= \int_0^{\frac{\pi}{2}} 35 \sin^4 x (1-\sin^2 x) \cos x \, dx$

$u = \sin x$

$du = \cos x \, dx$

$x=0 \quad u=0$

$x=\frac{\pi}{2} \quad u=1$

$= \int_0^1 35 u^4 (1-u^2) \, du$

$= \int_0^1 35 (u^4 - u^6) \, du$

$= 35 \left(\frac{1}{5}u^5 - \frac{1}{7}u^7 \Big|_0^1 \right)$

$= 35 \left(\frac{1}{5} - \frac{1}{7} \right) = 7 - 5 = 2$ //

$$\int \sin^4 x \cos^2 x \, dx$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)^2 \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) dx$$

$$= \frac{1}{8}\int (1 - \cos 2x)^2 (1 + \cos 2x) \, dx$$

$$= \frac{1}{8}\int (1 - 2\cos 2x + \cos^2 2x)(1 + \cos 2x) \, dx$$

$$= \frac{1}{8}\int 1 - \cos(2x) - \cos^2(2x) + \cos^3(2x) \, dx$$

$$= \frac{1}{8}\int (1 - \cos(2x)) \, dx - \frac{1}{8}\int \cos^2(2x) \, dx + \frac{1}{8}\int \cos^3(2x) \, dx$$

$\downarrow$ etc...

$\downarrow$

$$\int \frac{1}{2} + \frac{1}{2}\cos(4x) \, dx$$

$\downarrow$ etc...

$\downarrow$

$$\int \cos^3(2x)\,dx = \int \cos^2(2x)\,\cos(2x)\,dx$$

$$= \frac{1}{2}\int (1-\sin^2(2x))\,2\cos(2x)\,dx$$

$u = \sin(2x)$

$du = 2\cos(2x)\,dx$

$$= \frac{1}{2}\int (1-u^2)\,du \longrightarrow \text{etc} \ldots$$