11.10 Applications of Power Series

Fact:

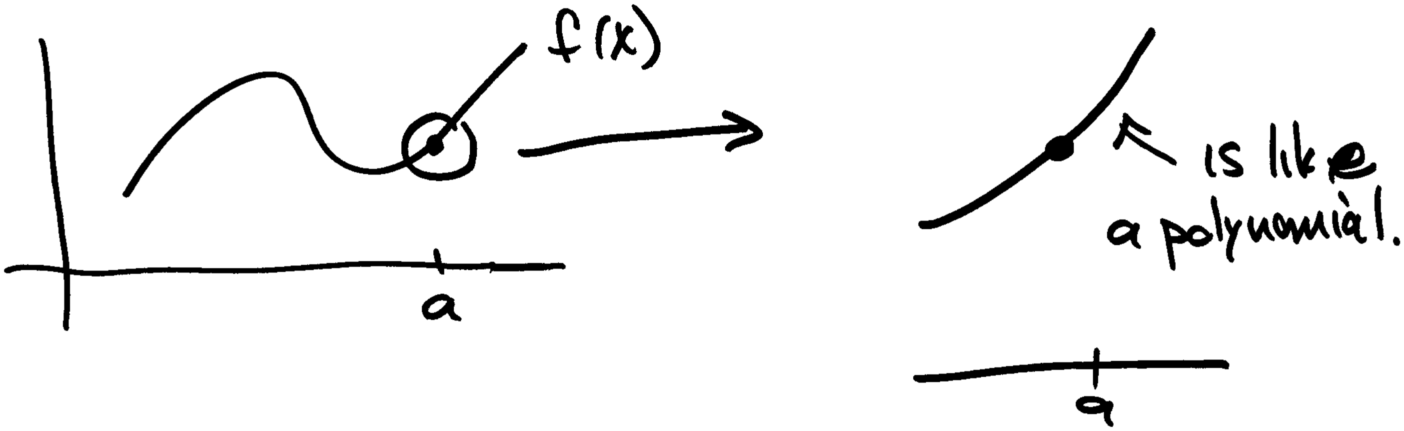
$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

true in all cases studied here.

$f(x)$ is close to its Taylor polynomials, i.e. partial sums of its Taylor series, when x is near a (center of series).

This says:

$$\boxed{\text{Any function is locally a polynomial}}$$



1. Binomial series

Consider Taylor series for $f(x) = (1+x)^m$ (where m is any number) at $x = 0$.

$$f(x) = (1+x)^m$$
$$f'(x) = m(1+x)^{m-1}$$
$$f''(x) = m(m-1)(1+x)^{m-2}$$
$$f'''(x) = m(m-1)(m-2)(1+x)^{m-3}$$
$$\vdots$$
$$f^{(k)}(x) = m(m-1)(m-2)\cdots(m-k+1)(1+x)^{m-k}$$

Evaluate at $x=0$.

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots + \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}x^k + \cdots$$

$$\boxed{(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k \quad \text{where} \quad \binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}}$$

This holds when $|x| < 1$

e.g

$$\binom{m}{0} = 1, \quad \binom{m}{1} = m, \quad \binom{m}{2} = \frac{m(m-1)}{2}, \quad \binom{m}{3} = \frac{m(m-1)(m-2)}{6}$$

etc...

e.g. #2.

$$(1+x)^{1/3} \cong 1 + \frac{1}{3}x + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2}x^2 + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{6}x^3$$

$$\cong 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3$$

$$\frac{\cancel{10}\,5}{27}\cdot\frac{1}{\cancel{6}\,3}$$

#6) $$\left(1-\frac{x}{2}\right)^{-2} = \left(1+\left(-\frac{x}{2}\right)\right)^{-2}$$

$$\cong 1 + (-2)\left(-\frac{x}{2}\right) + \frac{(-2)(-3)}{2}\left(-\frac{x}{2}\right)^2 + \frac{(-2)(-3)(-4)}{6}\left(-\frac{x}{2}\right)^3$$

$$= 1 + x + \frac{3}{4}x^2 + \frac{1}{2}x^3$$

In this case the Taylor series will converge for $\left|-\frac{x}{2}\right| < 1$ or $|x| < 2$

$$\frac{|x|}{2} < 1$$

2. Approximations to integrals

Rcl: $\int e^{x^2}dx$ has no "closed-form" solution.

We can compute it as a power series.

$$e^t = \sum_{n=0}^{\infty} \frac{1}{n!} t^n$$ holds for all t.

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{1}{n!}(x^2)^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}$$

$$\int e^{x^2}dx = \int \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}\, dx$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int x^{2n}\, dx = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{2n+1} x^{2n+1}$$

Also converges for all x.

e.g. $\int \cos(t^4)\, dt$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$\cos(t^4) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(t^4)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{8n}$$

$$\int \cos(t^4)\,dt = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{8n}\,dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \int t^{8n}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!(8n+1)} t^{8n+1}$$

3. Euler's formula. $\quad i^2 = -1$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^t = \sum_{n=0}^{\infty} \frac{1}{n!} t^n \qquad \cos(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n}$$

$$\sin(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n+1}$$

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{1}{n!}(i\theta)^n = \sum_{n=0}^{\infty} \frac{1}{n!} i^n \theta^n$$

$i^0 = 1 \quad i^1 = i \quad i^2 = -1 \quad i^3 = i^2 \cdot i = -i$

$i^4 = (i^2)^2 = (-1)^2 = 1 \quad i^5 = i \quad i^6 = -1$ etc $\cdots$

$$= \underbrace{\sum_{m=0}^{\infty} \frac{1}{(2m)!} i^{2m} \theta^{2m}}_{\text{even } n} + \underbrace{\sum_{m=0}^{\infty} \frac{1}{(2m+1)!} i^{2m+1} \theta^{2m+1}}_{\text{odd } n}$$

$$= \sum_{m=0}^{\infty} \frac{1}{(2m)!} (i^2)^m \theta^{2m} + \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} i \cdot (i^2)^m \theta^{2m+1}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \theta^{2m} + i \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \theta^{2m+1}$$

$$= \cos\theta + i \sin\theta$$

THE END